FABER-CASTELL

MATHEMA

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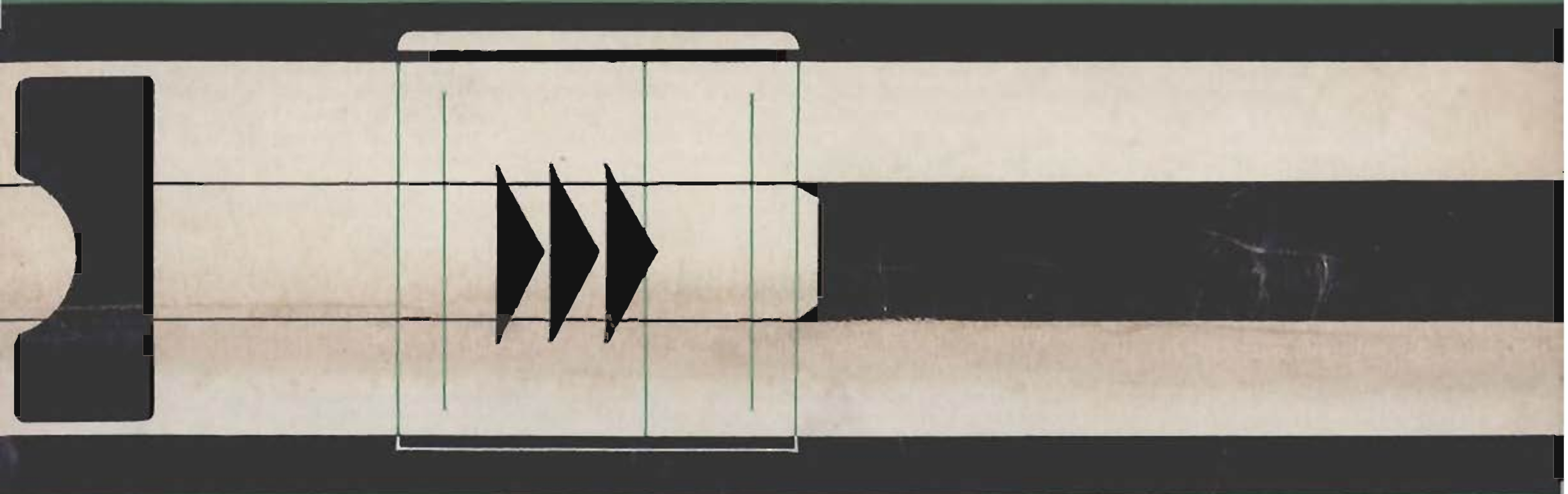
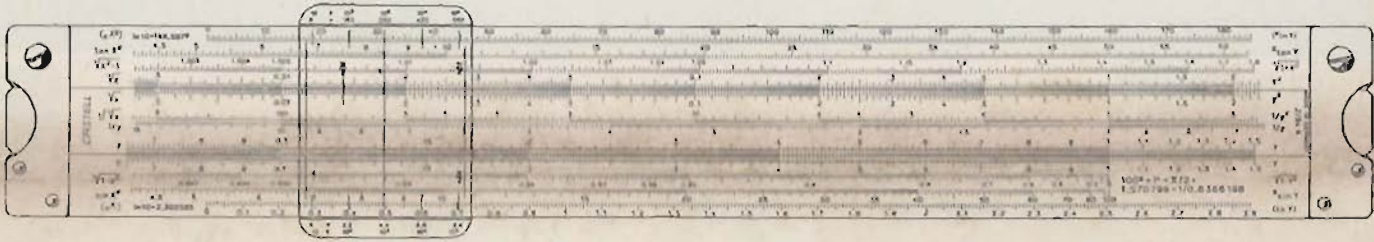
Universal Slide Rule

System Dr.-Ing. Moeller





The MATHEMA Universal Logarithmic Slide Rule

by Dr. Ing. Eugen Moeller, Darmstadt.

History of the Logarithmic Slide Rule

The logarithmic slide rule was invented and developed mainly as a result of

the formulation of logarithms by **Jost Bürgi** (1607) and **Lord John Napier** (1614),
the plotting of logarithmic scales by **Edmond Gunter** (1624),
the adoption of a second sliding ruler by **Wingate** (1627),
the arrangement of two similar logarithmic rules by **William Oughtred** (1630),
the invention by **Seth Partridge** of the slide running in a stock (1657),
the formulation of double logarithmic scales by **Roget** (1815),
the re-invention of the cursor by **Mannheim** (1851)

and the classical collaborative work "Darmstadt" under the direction of **Prof. Alwin Walther** (1934).

Construction of the Mathema Slide Rule

The **Mathema Slide Rule** has been created for use in practical mathematics and for the mathematical treatment of the natural sciences. With such purpose in view this Slide Rule is constructed as a double-sided rule (Duplex type) and has been provided with scales of all the elementary functions to such a degree of completeness that calculation can be carried out in as direct a manner as possible, thus not only making these calculations simple to achieve but also ensuring maximum accuracy. The manipulation of the Mathema Slide Rule is facilitated in no small degree by the arrangement on the front of the Rule of the scales which are most frequently used, the introduction of a common unit for the arguments of circular and hyperbolic functions, the designation according to formula of the primary functions and their inversions, the logical numbering of the scales and the long length of the scales and the markings on the cursor.

The **stationary principal scale,** to which all the other scales on the stock are related, is designated by Y.

The **movable principal scale,** to which all the other scales on the stock are related, is designated by y.

For easier handling, the two main scales are provided both on the front (lower slide and stock) and on the back (upper slide and stock); they are specially marked in the decade section (0·1-10).

The primary functions are designated by f(X) or f(x), according to whether they are based on principal scale Y or principal scale y. Here the usual arrangement applies:

$$Y = f(X)$$
$$y = f(x)$$

The **inverse functions** start from the principal scales Y and y are designated by f(Y) or f(y).

In order to make it possible to represent the functions in the range which is of practical importance, **some scales are plotted in several stages.**

It is for this same purpose that in the case of many of the functions there are scale extensions beyond the principal decade of the principal scales.

In the case of some of the functions, the **repetitions of the scale** which also extend beyond the principal decade of the principal scales make it unnecessary in many cases to change the position of the slide from one terminal position to the other.

All the scales running from left to right are in **black.** The **retrogressive scales,** that is to say those whose numbers increase from the right towards the left of the slide rule, are in **red.**

The **common unit** used for the arguments of the circular and hyperbolic functions, which are related to one another and are therefore often coupled with one another in equations and formulae, is the metric degree. For the easy conversion of metric degrees into circular measure (radians), the cursor is provided with setting marks for the factor $\pi/2$ on the normal scales.

The **positions of the decimal point** in the figures given on the Mathema Slide Rule are **uniformly based** on the trigonometrical and Pythagorean scales. This does not apply to the scales with functional expressions in brackets.

For the sake of easy legibility, the designations of the numerical values have been kept short.

A **point** preceding or following a figure signifies that so many noughts should precede or follow this figure as indicated by the small whole-number power of ten index given near by.

The scales, except those for e^x or log Y are **divided irregularly.** In particular, the intervals between the marks on the scale at different parts of a given scale are of different lengths. Units in the decimal system are not always divided into 10 intervals, but according to special requirements, either into 5 or 2. The mutual correspondences of the scale values of the various functions are generally established by means of the principal hair line of the **cursor.** At the left-hand end of the Slide Rule it is an advantage to use the left hair line of the cursor and at the right-hand end of the Slide Rule to use the right hair line of the cursor, if the e^x or log Y scales are not being used. The distances of the hair lines on the cursor from one another correspond to the factors $\pi/2$, and $\pi/4$.

The paragraphs giving **examples of computations** in this booklet and on the back of the Mathema Slide Rule are arranged from left to right in the order of the individual computation stages and therefore in general do not agree with the spatial distribution of the values on the scales. The functional scales required for the examples are given as partial reproductions of the inscriptions.

The Theory of Slide Rule Computation

The principle of computation using the logarithmic slide rule consists in converting the addition and subtraction (which can be carried out mechanically) of sections of the rule into the two higher processes of calculation, by virtue of the fact that these sections of rule represent the functional values in question on a logarithmic scale.

We get the **multiplication** and **division** of the natural numbers, if the logarithms of the natural numbers are plotted on the two scales which are used in conjunction with one another. Thus,

$$\log a + \log b = \log a \times b$$
$$\log a - \log b = \log a \div b$$

The starting point and finishing point of the simple logarithmic scale is the number 1, because 1 is a factor which has no effect on other numbers and $\log 1 = 0$.

The scales with the logarithms of the natural numbers are the **normal scales** of the Slide Rule; these are the principal scales in relation to the other scales.

Powers and **roots** of the natural numbers are obtained if one of the scales is a principal scale and the other is a scale of double logarithms of the natural numbers. Thus,

$$\log(\log a) + \log n = \log(n \times \log a) = \log(\log a^n),$$
$$\log(\log a) - \log n = \log\left(\frac{1}{n} \times \log a\right) = \log(\log a^{\frac{1}{n}}).$$

The starting point and finishing point of the double logarithmic scale is the base of the logarithms, as their logarithm $= 1$ and the logarithm of $1 = 0$.

A reversal of the process of calculation gives the **exponents:**

$$\log(\log a^n) - \log(\log a) = \log(\log a^n \div \log a) = \log n$$
$$\log(\log a^{\frac{1}{n}}) - \log(\log a) = \log(\log a^{\frac{1}{n}} \div \log a) = \log \frac{1}{n}$$

The exponents n and $1/n$ are also the logarithms to base a of the antilogs a^n and $a^{\frac{1}{n}}$ respectively.

Generally speaking there is no need for any **extension** of the calculation

$$\log(\log a) + \log(\log b) = \log(\log a^{\log b}) = \log(\log b^{\log a})$$

It is sufficient if computations of this and similar types can be carried out on the Slide Rule by means of simple steps.

On the other hand, in order to increase the possibilities of using the Slide Rule in mathematics, physics and engineering, it is important that the **elementary functions** as well as certain other functions should be included on the Slide Rule and shown in relation to the principal scales. With the scales of functions f(X) and f(x) it is possible by projecting them on to the principal scales or on to the scale of double logarithms to carry out all the higher mathematical operations which correspond to the following relationships:

$$\log f(X) + \log f(x) = \log (f(X) \times f(x)),$$
$$\log f(X) - \log f(x) = \log (f(X) \div f(x)),$$
$$\log (\log f(X)) + \log f(x) = \log (\log f(X)^{f(x)}),$$
$$\log (\log f(X)) - \log f(x) = \log (\log f(X)^{1/f(x)}).$$

The **relations between the scales of functions** are also of importance. By transferring from a scale f(X) to a scale f(Y), the function $f(X)_{f(y)}$ is found, because the value Y is replaced by the function f(X).
The same applies to transferring from a scale f(x) to a scale f(y).
When transferring from a scale f(X) to a scale f(y), f(X) still requires the factor which is shown opposite the end calibration Y = 1 on the principal scale.
A corresponding state of affairs obtains when transferring from a scale f(x) to a scale f(Y). The factor for f(x) lies opposite y = 1 on Y.
By making use of the marks on the cursor it is also possible to introduce π factors when transferring in this way.
On account of the co-ordination of the functional scales with the principal scales, they too are in logarithmic form. Nevertheless, the general fact that they are logarithms is not shown in the designations and the figures marked on the scales, but for the sake of simplicity the antilogs are printed direct by the calibrations of the scales. The fact that we are dealing with a **logarithmic** slide rule is quite sufficient to remind us of the true state of affairs.

The functions are included on the Slide Rule in **continuous** form. This means that they are **easy to interpolate** but can only be set and read off with a limited degree of accuracy. If L is the length on the slide rule which is used to represent the unit log e = 1, then the length z of the logarithmic section of rule of size y is:

$$z = L \times \log y.$$

When setting and reading errors amount to the length $\triangle z$, the relative error of the principal scale y is:

$$\triangle y/y = \triangle z/L;$$

it is therefore constant and independent of the functions f(x) which may have been transferred on to the scale y.
If L = (decade length of 200 mm) /log 10 = 86·8 . . mm and $\triangle z$ = L/500 = 0·17 . . mm., we get the relative error on the principal scale y as 2‰.

If n multiplications or divisions follow one another, the apparent error will grow to $\sqrt{n}$ times as much, in accordance with the Gaussian law of errors.
The length z of the logarithmic section of rule of a function f(x) related to the principal scale is

$$z = L \times \log f(x).$$

If the setting and reading errors amount to a length Δz, the **relative error on the scale of functions f(x)** is

$$\Delta x/x = \Delta z\, f(x)/Lx\, f'(x).$$

The errors expressed in ‰ are shown in the following table for when $\Delta z/L = 1/500$:

f(y)	y = 0·01	0·1	1	10	f(y)
x		2	2		y
$1/x$		—2	—2		$1/y$
$1/\sqrt{x}$		—4	—4		$1/y^2$
$\sqrt{x}$		4	4		y^2
$\sqrt{1-x^2}$		—0·02	—∞		$\sqrt{1-y^2}$
$\sqrt{x^2-1}$		0·02	1		$\sqrt{1+y^2}$
sin x		2·03	∞		arc sin y
tan x		1·99	1·27		arc tan y
sinh x		1·99	1·61	0·66	arc sinh y
cosh x			∞	0·67	arc cosh y
tanh x		2·03	∞		arc tanh y
e^x			∞	0·87	e^y
log x	0·02	0·2	2	20	log y

In describing the calculation process a simple scheme has been employed, in which values on adjacent scales are marked with one oblique stroke (/) and the transition from one scale section to another with two oblique strokes (//). The scales are indicated by placing the function f(x) in front of the letters Sc; the setting or reading value is shown subsequently. The letters b. p. mean the adjustment of the slide to the basic (zero) position, in which the main scales coincide.

The Scales of the Mathema Slide Rule and their Relationship to one another.

The Principal Scales X and y and their reciprocals 1/y

The sections corresponding to a decade from 0·1 to 1 and from 1 to 10 etc. on a logarithmic scale of the natural numbers repeat one another exactly, because the logarithms only differ by the figure of the power of ten. One decade section of the principal scales Y and y therefore already contains all possible numbers, if we disregard the place values. Use is made of this in that each calibration for a whole number power of ten is regarded as the starting or finishing calibration of the principal scale. This means that the slide must be transposed in relation to the principal scales from one position to the one which is congruent with it at the other end of the rule, so that the result of the computation will be brought within the convenient range of the slide rule.

The scale of $^1/y$ is a retrogressive principal scale according to the relationship log $^1/y$ = — log y.

This scale makes it possible to **convert** a multiplication by a into a division by $^1/a$ and vice versa.

The relationships between the principal scales Y on the one hand and y and $^1/y$ on the other are those of **multiplication** and **division.**

As a rule, single computations are begun by **moving the cursor,** so that when the slide has been moved the result will be given **within the principal decade** and so that in the case of longer expressions it is possible **to continue the computation immediately.**

In the case of **tabular calculations** it is a good plan to set one end marking of the slide opposite to the constant term on scale Y so that it is possible either to multiply by variable factors by means of scale y or else to divide by variable divisors by means of scale $^1/y$.

The setting of the numbers on the principal scales, which can be carried out with the help of the hair lines on the cursor or in the case of scales Y and y by means of a terminal calibration of the other scale, is effected without regard to the position of the decimal point within the sequence of digits. In the case of numbers such as 1234, the 4th digit can only be set by estimation. In the case of figures such as 8765 it is hardly possible to give sufficient attention to the fourth digit.

A **division** such as 2468 ÷ 8·765 = 281·6 is carried out by placing the hair line of the cursor on Y = 2468 and then bringing y = 8765 on the slide underneath the hair line of the cursor. The result is found on scale Y opposite a terminal calibration of scale y; Y Sc 2468 / y Sc 8·765 // y Sc 1 / Y Sc 281·6.

A **multiplication** such as 234 × 567 = 1327 × 10^2 is carried out by setting the hair line of the cursor on Y = 234 and then bringing $^1/y$ = 567 on the slide underneath the hair line of the cursor. The result is found on scale Y opposite a terminal calibration of the slide; Y Sc 234 / $^1/y$ Sc 567 // y Sc 0·1 / Y Sc 1327 × 10^2.

The last places of simpler quotients and products can be given exactly by consideration. With these it is usual to estimate the scales. Examples: 605 ÷ 4 = 151·25. 202 × 3 = 606.

The **position of the decimal point** is found by a rough mental calculation or in cases which cannot be seen at a glance, by estimating the powers of ten. Example: 246800 ÷ 0·008765 = approx. 3 × 10^{5+2} = 2816 × 10^1.

By **alternating** the use of the cursor and the slide it is possible to carry out **continuous** multiplication and division. When this is done, there is no need to read off any of the intermediate results or to set the terminal calibration of the slide on to it. If, for instance, the numbers are all factors for multiplication, then scales $^1/y$ and y are used alternately in conjunction with scale Y. In this way, a × b × c × d becomes $a \div \frac{1}{b} \times c \div \frac{1}{d}$. It will be found that 12 × 34 × 56 × 78 = 1782 × 10^3. In the case of expressions such as $\frac{a}{bcd}$ it is found that $\frac{1782}{(12 \times 34 \times 56)}$ = 0·078.

The following is a diagrammatic representation of how the possible combinations of the second stage of the values a, b and c can be computed.

1/y		c	b	b c	1/y
y	b c	b	c		y
Y	a ac/b	a a/bc	a abc	a ab/c	Y

If one factor of a product is in the vicinity of 1 and is known to a very great degree of accuracy, as in the case of a few inverse functions on the Mathema Slide Rule, increased accuracy can be obtained in the multiplication by **decomposing** the factor into the figure 1 and the deviation from this figure. Example: $0{\cdot}9876 \times 543 = (1 - 0{\cdot}0124) \times 543 = 543 - 6{\cdot}73 = 536{\cdot}27$. By operating in the other way, only the figure 536 is obtained.

If a **quotient** obtained from accurately known terms is in the near vicinity of 1, or if the difference between these terms is known very exactly, this is also a case where **decomposition** is profitable. Example: $\frac{456}{455} = 1 + \frac{1}{455} = 1{\cdot}002198$.

By operating according to the other method, only the figure 1·002 is obtained.

Opposite the initial and terminal calibrations of scales Y and y will be found values which are reciprocals of one another. This makes it possible to work out fractions as **reciprocates**, with the numerator and denominator interchanged, so as to arrive at the results direct. This method is indicated if a primary function related to scale Y is situated in the denominator of a factor. Example: $678 \times \text{cosec}\ 30^g = 1/(\sin 30^g \div 678) = 1493$, and in carrying out this computation it is not necessary to read off $\sin 30^g = 0{\cdot}454$. The result now appears on scale y, the hair line of the cursor being placed over $\sin 30^g$ and 678 on the y scale of the slide being brought under the hair line.

By means of the $\pi/2$ mark on the cursor it is possible to convert metric degrees ^{d} of an angle into the circular measure in radians and vice versa. Thus

DEGREES — comparison only

(grad)

$1^d = \pi/200$ rad $= 0{\cdot}01570796$ rad, = 0.9°

1 rad $= 200/\pi^d = 63{\cdot}66198^d$ (grad) = 57.2957795°

$\pi = 3{\cdot}1415927 = 63{\cdot}66198^d$. (grad) = 57.2957795°

1° = 1.1111111 d (grad)

(Arguments for trig. & hyp. functions should be converted first by A to grades

The linear interpolation of numerical values depends on the sufficiently accurate proportionality between the differences in the arguments and the differences in the functional values in the cases in question. The Slide Rule can conveniently be used for this purpose even when four-figure tables are used. For repeated interpolations between the smaller number a and the larger number b, which are both only known with the accuracy of the slide rule, and the difference between arguments d, the following method is suitable. Place d in the y scale opposite b—a on the Y scale. Then the number a on the Y scale will be opposite a number a on the y scale and also the number b on scale Y will be opposite the number c + d on scale y; the changes in the argument which can be seen on scale y, i.e. the amounts which vary from c or from c + d as the case may be, give the relative functional values on scale Y and vice versa. We get the following scheme:

y	d	c		c+d	y
Y	b-a	a	<	b	Y

This may be proved by writing it out in full as follows:

$$c = ad/(b-a)$$
$$c + d = bd/(b-a) = ad/(b-a) + d$$

The value of c can be rounded off without any appreciable ill-effect, in order to simplify reading off the changes in argument.

Example: a = 456; b = 789; d = 2. It is found that c = 2·7388, which can be approximated to 2·74. For the argument increase of 0·234 the functional value is 495, if the arguments increase from a to b; otherwise the functional value is 750.

The roots of the quadratic equation

$$x^2 + ax + b = 0$$

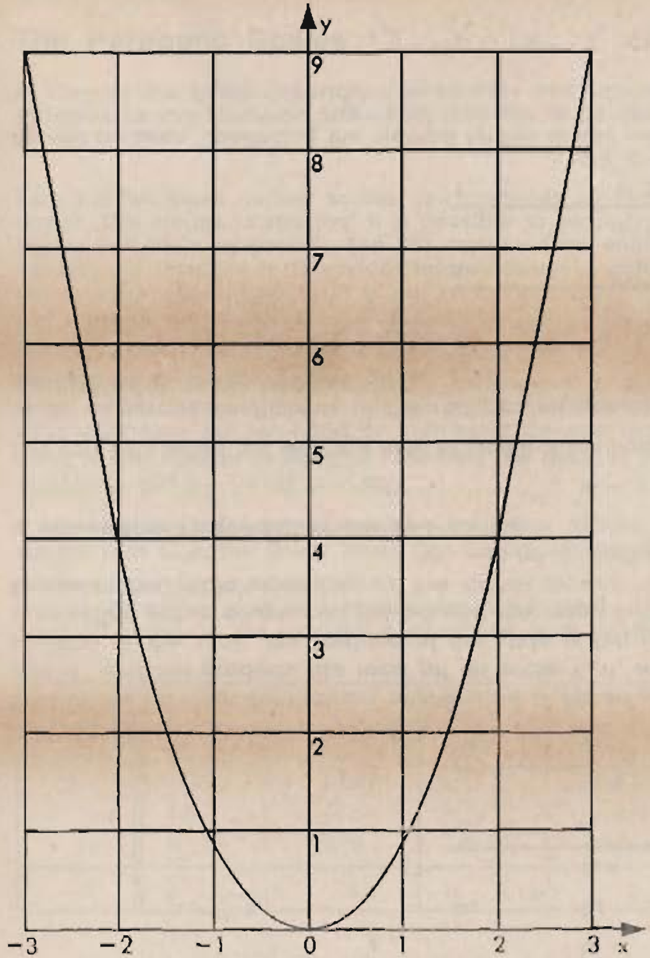
can be obtained in real numbers from the abscissae of the points of intersection of the unit parabola

$$y_1 = x^2$$

and the straight line

$$y_2 = -ax - b.$$

It is sufficient to lay the edge of a ruler on the parabola and make an estimation, if the solutions found are improved by means of the Slide Rule.



Quadratic Unit Parabola $y = x^2$

For this purpose we put the equation we have just mentioned

In slide rule form: $x + \frac{b}{x} = -a$

and we then set one terminal calibration of the slide over b on scale Y; by means of the hair line of the cursor it is then possible, working from scale Y, to find on scale $1/y$ the value b/x pertaining to any desired value of x. The sum of the two values should be $-a$, which must be checked.

As the roots x_1 and x_2 obey Vieta's Law, according to which

$$x_1 + x_2 = -a$$

both the roots will be found on the slide rule at the same time.

Example: $x - \frac{2}{x} = 3;\ x_1 = 3{\cdot}56;\ x_2 = -0{\cdot}56.$

If the straight line $y/2$ does not intersect the parabola y, and the roots are therefore conjugate complex, the following formula, which is also valid, is used:

$$x_1, x_2 = -a/2 + \sqrt{a^2/4 - b}.$$

The criteria of the co-ordinates of the diagram of the quadratic unit parabola on the left may be used as numerations if it is necessary to vary the scales of the co-ordinates. When doing this, care must be taken to see that the equation of the quadratic unit parabola is maintained. If, for instance, the size of the abscissa is doubled, the size of the ordinate must be quadrupled.

The principal scales are suitable for the use of **Horner's Method** for the determination of the value of equations of the following type: $b_0 = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

for a given value of x. For this purpose, the coefficients are written in a series and the system is completed step by step as shown, when $b_4 = a_4$, $b_3 = a_3 + b_4 x$, $b_2 = a_2 + b_3 x$ etc. By repeated application it is found that $c_1 = f'(x)/1!$, $d_2 = f''(x)/2!$ etc. for the accepted value of x.

		a_4	a_3	a_2	a_1	a_0
	↗		$+b_4 x$	$+b_3 x$	$+b_2 x$	$+b_1 x$
x		b_4	b_3	b_2	b_1	$b_0 = f(x)/0!$
			$+c_4 x$	$+c_3 x$	$+c_2 x$	
		c_4	c_3	c_2	$c_1 = f'(x)/1!$	
			$+d_4 x$	$+d_3 x$		
		d_4	d_3	$d_2 = f''(x)/2!$		

One of the terminal calibrations of the slide is set on x onthe scale Y. The hair line of the cursor is then set on b , c and d and the products are read off on scale Y. If, when doing this, one or the other factor should require the transposition of the slide, the position of the slide is left unchanged and the multiplication is carried out with the factor in question either doubled or halved and the result is then accordingly halved or doubled as the case may be.

Example: The cubic equation

$$x^3 - 15{\cdot}4x^2 + 82{\cdot}65\,x - 150{\cdot}7 = 0$$

is relieved of its quadratic term by inserting

$$x = y + \frac{15{\cdot}4}{3}$$

This can be carried out with the help of Horner's Method by carrying it out in two stages with the figure $\frac{15{\cdot}4}{3} = 5{\cdot}1333$, so as to get the coefficients of the reduced equation. The value 3 on scale y is set opposite 154 on scale Y without the help of the cursor and we get

		1	—15·4	82·65	—150·7
15·4/3	↗		+ 5·133	—52·70	+153·7
		1	—10·267	29·95	3·0
			+ 5·133	—26·35	
		1	— 5·133	3·60	

As will be seen, continuing the method further would cause the second term to disappear. The reduced equation is:

$$y^3 + 3{\cdot}6\,y + 3 = 0.$$

The Parabolic Scales $\sqrt{X}$ .. Y^2, $\sqrt{x}$ .. y^2 and their Reciprocals $1/\sqrt{x}$ .. $1/y^2$

In view of the great importance of squares and square roots, the parabolic scales and their reciprocals have been included on the Mathema Slide Rule with the same degree of completeness as the principal scales. As

$$\log \sqrt{x} = \frac{\log x}{2}$$

here too we have normal scales; the divisions on these scales, however, are half the size of those of the principal scales. This means to say that it is possible to multiply and divide with them in just the same way as with the principal scales and their reciprocals and this can be done entirely without having to transpose the slide, which is very convenient, for instance in the **linear interpolation** of tabular values. However, one has in this case to put up with double the relative error, if the root is not to be extracted from the product or quotient.

As both the square roots and squares of the ordinary numerical series represent a parabola in diagrammatic form, the parabolic scale is a collective conception for the square root and square scale.

Two decades of the parabolic scale correspond to the decade of a principal scale. When extracting square roots, which can be done either by means of the hair line of the cursor or one of the terminal calibrations of the slide, one must start from the left-hand or right-hand decade according to whether the number of digits preceding the decimal point or the number of noughts following the decimal point is odd or even. Examples: $\sqrt{123} = 11{\cdot}9$; $\sqrt{12{\cdot}3} = 3{\cdot}507$; $\sqrt{0{\cdot}0123} = 0{\cdot}1109$; $\sqrt{0{\cdot}123} = 0{\cdot}3507$.

If in **compound expressions,** besides the linear factors and divisors there occur only those in the square or under the square root sign, the linear terms can be calculated either by means of the parabolic scales or the principal scales.

In the following table the results are shown for the case where a value a is set on scale Y by means of the hair line of the cursor, a value b on the respective scales of the slide is brought under the hair line and the results are read off opposite the terminal calibrations both on the stationary and movable principal and parabolic scales.

$\sqrt{X}$	a^2	a^2/b^2	1	a^2	$a^2 \times b^2$	1	a^2	$a^2 \times b$	1	a^2	a^2/b	1	Y^2
$\sqrt{x}$	b^2	1	b^2/a^2	$1/b^2$	1	$1/a^2b^2$	$1/b$	1	$1/a^2b$	b	1	b/a^2	y^2
$1/\sqrt{x}$	$1/b^2$	1	a^2/b^2	b^2	1	a^2b^2	b	1	a^2b	$1/b$	1	a^2/b	$1/y^2$
$1/y$	$1/b$	1	a/b	b	1	ab	$\sqrt{b}$	1	$a\sqrt{b}$	$1/\sqrt{b}$	1	$a/\sqrt{b}$	$1/y$
y	b	1	b/a	$1/b$	1	$1/ab$	$1/\sqrt{b}$	1	$1/a\sqrt{b}$	$\sqrt{b}$	1	$\sqrt{b/a}$	y
Y	a	a/b	1	a	ab	1	a	$a\sqrt{b}$	1	a	$a/\sqrt{b}$	1	Y

The next table corresponds to the preceding one, except that in this case a is on the $\sqrt{X}$ scale.

$\sqrt{X}$	$\underline{a}$	a/b^2	1	$\underline{a}$	ab^2	1	$\underline{a}$	ab	1	$\underline{a}$	a/b	1	Y^2
$\sqrt{x}$	b^2	1	b^2/a	$1/b^2$	1	$1/ab^2$	$1/b$	1	$1/ab$	$\underline{b}$	1	b/a	y^2
$1/\sqrt{x}$	$1/b^2$	1	a/b^2	b^2	1	ab^2	$\underline{b}$	1	ab	$1/b$	1	a/b	$1/y^2$
$1/y$	$1/b$	1	$\sqrt{a/b}$	$\underline{b}$	1	$\sqrt{ab}$	$\sqrt{b}$	1	$\sqrt{ab}$	$1/\sqrt{b}$	1	$\sqrt{a/b}$	$1/y$
y	$\underline{b}$	1	$b/\sqrt{a}$	$1/b$	1	$1/\sqrt{ab}$	$1/\sqrt{ab}$	1	$1/\sqrt{ab}$	$\sqrt{b}$	1	$\sqrt{b/a}$	y
Y	$\sqrt{a}$	$\sqrt{a/b}$	1	$\sqrt{a}$	$\sqrt{ab}$	1	$\sqrt{a}$	$\sqrt{ab}$	1	$\sqrt{a}$	$\sqrt{a/b}$	1	Y

The upper **mark** $\pi/4$ on the cursor, which counts from the principal line = 1 like the upper mark $\pi/2$, makes it possible to set the **area of a circle** of given diameter direct from a principal scale on to the corresponding parabolic scale or, conversely, to set the **diameter of a circle** of given area. Examples: $\pi/4 \times 2345^2 = 432 \times 10^4$ $\sqrt{2345 \times 4/\pi} = 54{\cdot}55$.

The **cube** and the square root of this and also the **cube root** and the square of this can be calculated by means of the principal and parabolic scales in such a way that they appear on stationary normal scales ready for immediate continued calculation.

In the following representation of the formation of the **3rd power** it is immaterial on which decade of the reciprocal parabolic scale the initial value is set. When computing the square roots of these, however, the decade rules for the extraction of square roots must be followed at once when setting the initial figures, because these terms are enclosed within brackets. Examples: $2^{3/2} = 2{\cdot}828$; $20^{3/2} = 89{\cdot}4$.

When forming the **cube root,** the fact is that it must be found by **trial and error**, symbolised by a double line. The values on scales $1/\sqrt{x}$ .. $1/y^2$ and Y under the cursor must agree. When doing this it is immaterial which decade of the stationary parabolic scale is used for setting the initial figure, provided the approximate answer is estimated first, so that it is possible to select the correct one from the three possible positions of the cursor. In the case of the square roots, the corresponding decade rules must be observed when setting the initial figures, because these terms are enclosed in brackets.

Examples: $2^{2/3} = 1{\cdot}587$; $20^{2/3} = 7{\cdot}37$; $200^{2/3} = 34{\cdot}2$.

$\sqrt{X}$		a^3	1	$\underline{b}$	$b^{2/3}$	1	Y^2
$\sqrt{x}$		1	$1/a^3$	1	$1/b^{1/3}$	$1/b$	y^2
$1/\sqrt{x}$	$\underline{a}$	1	a^3	1	$b^{1/3}$	b	$1/y^2$
$1/y$		1	$(a^{3/2})$	1	$(b^{1/6})$	$(\sqrt{b})$	$1/y$
y		1	$(1/a^{3/2})$	1	$(1/b^{1/6})$	$(1/\sqrt{b})$	y
Y	$\underline{a}$	$(a^{3/2})$	1	$\sqrt{b}$	$b^{1/3}$	1	Y

Fourth powers are set on the stationary parabolic scale by means of one terminal calibration of the slide, if the base is marked on scale Y by means of the hair line of the cursor and the slide is moved so that the base on scale $1/y$ comes underneath the hair line. In this way the square of the base has been squared in relation to the stationary parabolic scale.

Fourth roots are obtained without trial and error by extracting the square root twice over, and when doing this the relevant decade rules must be taken into account. Examples: $2^{1/4} = 1{\cdot}189$; $20^{1/4} = 2{\cdot}115$; $200^{1/4} = 3{\cdot}760$; $2000^{1/4} = 6{\cdot}69$.

The roots of the **reduced cubic equation**

$$x^3 + ax + b = 0$$

are obtained in real values from the abscissae of the points of intersection of the cubic unit parabola

$$y_1 = x^3$$

and the straight line

$$y_2 = -ax - b.$$

Laying the edge of a ruler on the cubic parabola and making an estimate is sufficient, if the solutions so obtained are corrected by means of the slide rule. For this purpose the equation we have given is converted into its slide rule form:

$$x + \frac{b}{x} = -a.$$

and one terminal calibration of the slide is set against b on the scale Y. By means of the hair line on the cursor one then obtains, starting from scale Y, the values of $\frac{b}{2}$ on scale $^1/y$ pertaining to every desired value of x^2 on the fixed parabolic scale.

$\sqrt{X}$		x_1^2	Y^2
$^1/y$		b/x_1	$^1/y$
y	1		y
Y	b	x_1	Y

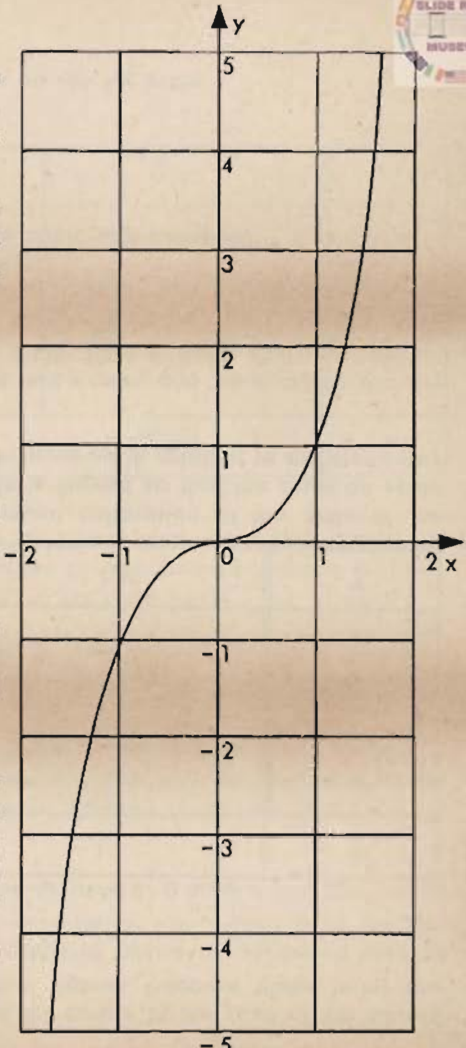
The sum of the two values must equal $-a$, and this has to be achieved by trial and error. If one root x_1 is known, the other two can be found from

$$x_2, x_3 = \frac{x_1}{2} \pm \sqrt{-a - \frac{3x_1^2}{4}},$$

which is particularly necessary, if these are complex conjugated. If two roots of the equation are known, the third is obtained from Vieta's Law, by completing the sum of the roots to the negative coefficient of the second highest term of the equation, in this case therefore to zero.

Example: $x^3 + 3{\cdot}6\,x + 3 = 0$; $x_1 = -0{\cdot}726$; $x_2, x_3 = 0{\cdot}363 \pm 2\,i$.

The measurement coefficients of the co-ordinates in the adjacent diagram of the cubic unit parabola can be used as numerations, if it is necessary to change the scales of the co-ordinates. When doing this it must be borne in mind that the equation of the cubic unit parabola must be maintained. This means that if the measurements of the abscissae are, say, doubled, the measurements of the ordinates must be multiplied by eight.



Cubic Unit Parabola $y = x^3$

The more accurate **extraction of the square root** from the number c is best carried out in the form

$$\pm \sqrt{c} = \pm \sqrt{a^2 \pm R} = a \pm x$$

as x can be determined by means of the slide rule in accordance with the following requirement:

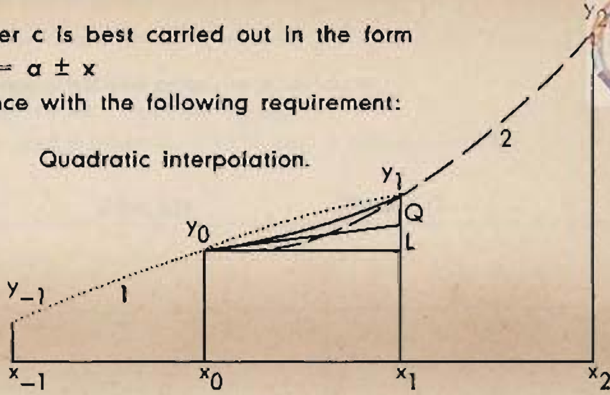
$$\frac{R}{(2a \pm x)} = x.$$

This process can also be applied after several stages of computation to extract the exact root.

Examples: $\sqrt{123456} = \sqrt{122500 + 956} = 350 + 1{\cdot}363 = 351{\cdot}363$
as $956 \div (700 + 1{\cdot}363) = 1{\cdot}363$.

$\sqrt{159876} = \sqrt{160000 - 124} = 400 - 0{\cdot}155 = 399{\cdot}845$,
as $124 \div (800 - 0{\cdot}155) = 0{\cdot}155$.

Quadratic interpolation.



In the quadratic interpolation of a function, the curve of the function is replaced by a parabola. Let the values of the function $y = f(x)$ be known for equal intervals between the arguments, and let us either look for the ordinate $y_0 = \triangle y$ for the abscissa $x_0 + \triangle x < x_1$ or else let us look for the abscissa $x_0 + \triangle x$ for the ordinate $y_0 + \triangle y < y_1$. The point $y_0 + \triangle y = f(x_0 + \triangle x)$ should now rest with a reasonable degree of accuracy on the substitute parabola; the value desired can then be determined by direct or inverse interpolation.

The substitute parabola must pass through both points x_0, y_0 and x_1, y_1. For the sake of simplicity its axis should be parallel to the ordinate axis. A suitable rule to act as a further condition for the parabola is that the ordinates of the substitute parabola should be the arithmetical mean of the ordinates of parabola 1 though the point x , y and parabola 2 through the point x_2, y_2.

If the linear interpolated term is L and the quadratic interpolated term is Q, then by definition

$$\triangle y = \frac{\triangle x}{x_1 - x_0} L + \left(\frac{\triangle x}{x_1 - x_0}\right)^2 Q$$

and $L = y_1 - y_0 - Q$.

For the inverse interpolation we get

$$\triangle x = \left(\sqrt{1 + 4 \triangle y Q / L^2} - 1\right) (x_1 - x_0)\, L / 2Q.$$

If the root expression is in the vicinity of 1 and can no longer be calculated with the Pythagorean scales, we find with the short form $k = \triangle yQ/L^2$,

that $\triangle x = (1 - k + 2k^2 - 5k^3 + 14k^4 - 42k^5 + 132k^6 - 429k^7 + 1430k^8 - . + . .)\,(x_1 - x_0)\,\triangle y/L.$

Q_1 of parabola 1 is found as follows:

$$y_0 - y_1 = L_1 + Q_1$$

$$y_1 - y_{-1} = 2L_1 + 4Q_1$$

$$Q_1 = (y_{-1} - 2y_0 + y_1) \div 2$$

In an analogous manner we find Q_2 of parabola 2 to be

$$Q_2 = (y_0 - 2y_1 + y_2) \div 2.$$

From which we get $Q = ((y_2 - y_1) - (y_0 - y_{-1})) \div 4.$

or, expressed in words: the quadratic interpolated term is one-quarter of the difference between the tabular difference following the interval in question and the tabular difference preceding it.

The substitute parabola is coincident with the curve given if this is a parabola of the type $y = x^2$; here $L = 2x$ and $Q = 1$ when the interval of the argument is 1, as will be seen immediately from the equation $(x + 1)^2 = x^2 + 2x + 1$. On the other hand, the substitute parabola does not agree with the parabola $y = \sqrt{x}$. This means that before drawing up a table of numbers one must test by prior quadratic interpolation to see which of the two variables is independent and permits of the greater accuracy, if there is any question of choosing between them. Very often it is advisable to set up one of the variables in the form of the reciprocal.

Example 1. Let us assume that $y = e^x$, where $x = 2{\cdot}30$ with intervals of 0·01. It is desired to find y when $x = 2{\cdot}315$ and x when $y = 10{\cdot}12$.

If we set this up as follows:

x	y	tabular diff.	Q	L
2·30	9·974 182			
		0·100 243		
2·31	10·074 425			
		0·101 249	0·000 506	0·100 743
2·32	10·175 674			
		0·102 268		
2·33	10·277 942			

we get $e^{2{\cdot}315} = 10{\cdot}074\,425 + 0{\cdot}050\,371 + 0{\cdot}000\,127 = 10{\cdot}124\,923$, which agrees with the true value.

Generally speaking, one sets $\frac{\Delta x}{x_1 - x_0}$ on scale Y, and then the linear part of Δy is read off with L on the scale y and the quadratic part of Δy with the slide in the same position and Q on scale $\sqrt{x} .. y^2$.

When y = 10·12, then k = 0·002 2722 and x = 2·314 514, which agrees with the true value.

Example 2. Given y = log x when x = 10·0 with intervals of 0·1, find y when x = 10·15 and x when y = 2·317.

If we set this up as follows:

x	y	tabular diff.	Q	L
10·0	2·302 585			
		0·009 950		
10·1	2·312 535			
		0·009 853	—0·000 0485	0·009 9015
10·2	2·322 388			
		0·009 756		
10·3	2·332 144			

we get log 10·13 = 2·312 535 + 0·004 951 — 0·000 012 = 2·317 474, which agrees with the true value.

If y = 2·317, k will = —0·002 2088 and x will = 10·145 194, in which the last figure is 1 too large. Both examples of inverse interpolation have been computed on the calculating machine, in order to demonstrate the accuracy of the quadratic interpolation in the cases in question. It must be acknowledged, however, that the Slide Rule can also prove a very useful aid in subsidiary calculations even when using methods which are more accurate in other respects.

The Pythagorean Scales $\sqrt{1-X^2}$ .. $\sqrt{1-Y^2}$ and $\sqrt{X^2-1}$ .. $\sqrt{1+Y^2}$

The two Pythagorean Scales include in conjunction with the main scale a graphical representation of the **ratios of the sides of right-angled unit triangles.**

The circular scale $\sqrt{1-X^2}$ .. $\sqrt{1-Y^2}$ in conjunction with the main scale for one co-ordinate of the unit circle gives the other co-ordinate and therefore the cosine for the sine of the corresponding angle, and vice versa. The co-ordinates of the unit circle are the sides of right unit triangles having a hypotenuse = 1, i.e. of hypotenuse unit triangles.

The **hyperbolic scale** $\sqrt{X^2-1}$ .. $\sqrt{1+Y^2}$ in conjunction with the main scale for one co-ordinate of the unit hyperbola gives the other co-ordinate and therefore the secant for the tangent of the corresponding angle, and vice versa. The co-ordinates of the unit hyperbola are one side and the hypotenuse of unit right triangles with the other side = 1, i.e. of other side unit right triangles.

As the Pythagorean scales and also the scales of transcendental functions must be broken off on one or both sides, extreme ratios cannot be calculated by means of the Slide Rule in a direct way. In these cases use is made of the approximation formulae obtained from the corresponding infinite series. When the values of x are small, a linear or quadratic relation between the independent and dependent variables is usually sufficient.

In many cases the section which is required of the slide when it has been moved extends over the section of Pythagorean (and even other) scales coupled with it. In such cases it is usually possible to get the desired result by first **doubling** or **halving** the given triangle and then halving or doubling the results as the case may be.

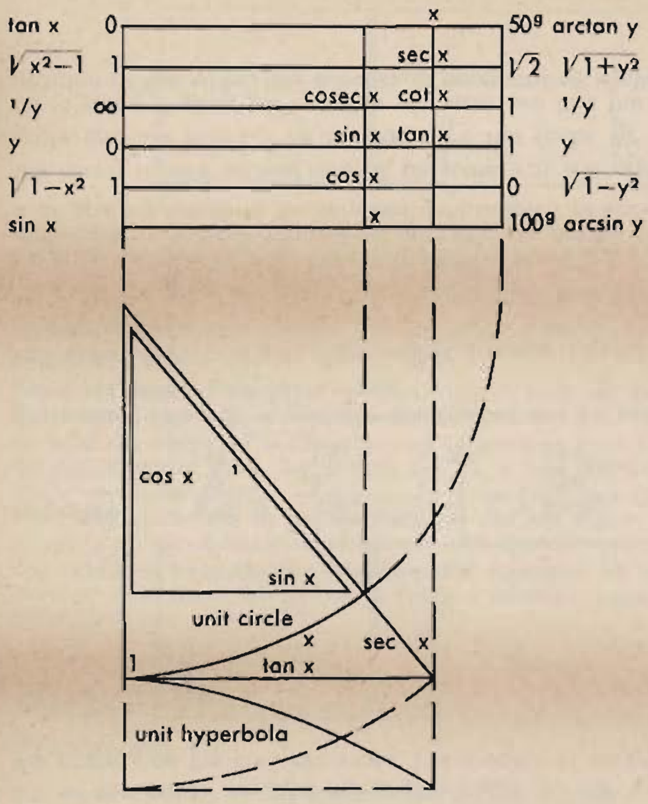
The following table shows the mutual relationship between the Pythagorean scales and the main scales when the slide is in its basic position, if the initial figure set on one of the scales is regarded as being z. The value of these relationships lies not only in their direct utilisability, but even more in the high degree of accuracy of the results.

$\sqrt{X^2-1}$	$\sqrt{2-z^2}$	$\sqrt{1+1/z^2}$	$\sqrt{1+1/z}$	$\sqrt{1+z}$	z	$\sqrt{1+Y^2}$
$\sqrt{x}$	$1-z^2$	$1/z^2$	$1/z$	z	z^2-1	y^2
$1/\sqrt{x}$	$1/(-z^2)$	z^2	z	$1/z$	$1/(z^2-1)$	$1/y^2$
$1/y$	$1/\sqrt{1-z^2}$	z	$\sqrt{z}$	$1/\sqrt{z}$	$1/\sqrt{z^2-1}$	$1/y$
$\sqrt{1-X^2}$	z	$\sqrt{1-1/z^2}$	$\sqrt{1-1/z}$	$\sqrt{1-z}$	$\sqrt{2-z^2}$	$\sqrt{1-Y^2}$

The Trigonometric and Arc-trigonometric scales sin X° .. arc sin Y and X° .. arc tan Y

REFER p. 9

The **trigonometric and arc-trigonometric scales** taken in conjunction with the principal scale contain the trigonometric functions which appear direct on the right-angled unit triangles.



Right-angled triangles in the unit circle and on the unit hyperbola, with equal angles. The scales in the diagram have not been rendered logarithmic.

On the slide rule **trigonometric functions** are preferred to angles because the functions can be projected on to the principal scale for the purpose of further computation, whilst the angles themselves can only be read off. This is justified in the fact that angles are not often required, but more frequently we require to know the results of certain angles. The same also applies to the argument of the hyperbolic functions.

Whereas in mathematics radian measure is the natural argument for angles, the **metric degree** or **the right angle** is the most suitable artificial argument for computations. The natural argument of the trigonometric and hyperbolic functions is often supplied with the factor $\pi/2$, 2π or π so that the right angle or a whole multiple is obtained when the π factor is removed. In the Mathema Slide Rule, therefore, the divisions for the said functions are shown in metric degrees.

The scale sin Xg .. $\frac{200}{\pi}$ x arc sin Y (abbreviated as gsin Y), is closely bound up with the circular scale. As a result of the common principal scale, $Y = \sqrt{1-X^2}$ = sin Xg. From this we get $\sqrt{1-Y^2}$ = cos Xg and the fact that the hair line on the cursor shows a set of three values for the two sides of the hypotenuse unit triangle and the relative angle.

The **side** a of any right-angled triangle having a hypotenuse c and the other side b is found from

$$a = \sqrt{c^2 - b^2} = c\sqrt{1 - \left(\frac{b}{c}\right)^2}$$

On the Mathema Slide Rule we produce the relationship between the sides of the given triangle and the sides of the hypotenuse unit triangle in accordance with the second of these equations. For this purpose, c on scale y is set opposite the terminal calibration 1 of scale Y, or if necessary opposite calibration 0·1. By setting the hair line of the cursor on b on scale y it will give us $\frac{b}{c}$ on scale Y, and in addition $\sqrt{1-(b/c)^2}$ on the circular scale and the angle β on the sin... arc sin scale. By transferring the amount $\sqrt{1-(b/c)^2}$ from the circular scale on to scale Y using the hair line of the cursor, we find the required side $\sqrt{c^2-b^2}$ on scale y without altering the position of the slide and we find the angle α on the sin.. arc scale. The process of computation is shown step by step in the following diagram:

y	c	b	$\sqrt{c^2-b^2}$	y
Y	1	b/c	$\sqrt{1-(b/c)^2}$ ↑	Y
$\sqrt{1-X^2}$	0	$\sqrt{1-(b/c)^2}$ ⟶		$\sqrt{1-Y^2}$
sin X^g	100	β	α	^{g}sin Y

Examples: c = 6·78; b = 4·56; a = 5·02; α = 53·09;
15 6 13·75 73·8
15 1·5 14·925 93·6

sin vers 40^g = 1 — cos 40^g = 0·1910
sem 40^g = (1 — cos 40^g)/2 = $\sin^2 20^g$ = 0·0955

Expressions like $\sqrt{d^2-c^2-b^2}$ can be worked out by repeating the process of calculation.
Examples: $\sqrt{987^2-654^2-321^2} = 666$.

The **hypotenuse** c of any right-angled triangle having other sides a and b can be found from:

$$c = \sqrt{a^2+b^2} = a\sqrt{1+(\frac{b}{a})^2}.$$

Let a be greater than b. On the Mathema Scale we produce the relationship between the sides of the given triangle and the sides of the 'other side' unit triangle on the basis of the second of these equations. For this purpose, we set a on scale y opposite the terminal calibration 1 of scale Y, or if necessary opposite calibration 0·1. If the hair line of the cursor is now placed over b on scale y it will also give us $\frac{b}{c}$ on Y, and in addition $\sqrt{1+(b/a)^2}$ on the hyperbolic scale and the angle β on the tan.. arc tan scale. By transferring the amount $\sqrt{1+(b/a)^2}$ from the hyperbolic scale on to scale Y using the hair line of cursor, we find the desired hypotenuse $\sqrt{a^2+b^2}$ on scale y without altering the position of the slide. This process of computation is shown step by step in the following diagram:

$\tan X^g$	50		β		$^g\tan Y$
$\sqrt{X^2-1}$	$\sqrt{2}$		$\sqrt{1+(b/a)^2}$		$\sqrt{1+Y^2}$
y	a	>	b	$\sqrt{a^2+b^2}$	y
Y	1		b/a →	$\sqrt{1+(b/a)^2}$	Y

Examples: a = 6·78; b = 4·56; c = 8·17; β = 37·7g
15 6 16·16 24·2
15 1·5 15·075 6·35

Expressions like $\sqrt{a^2+b^2+c^2}$ can be worked out by repeating the process of computation.

Example: $\sqrt{987^2+654^2+321^2} = 1227$.

Since the same position of the cursor gives both **sin Xg and cos Xg,** it is possible to pass from one of these values to the other according to the expressions cos arc sin Y and sin arc cos Y without first having to determine the angle. In addition either of the two values, if greater than $1/\sqrt{2}$ can be obtained more exactly on the circular scale than on the principal scale, by setting sin Xg = cos (100-Xg) or cos Xg = sin (100-Xg).

Since the same position of the cursor gives both **tan Xg and sec Xg,** it is possible to pass from one of these values to the other according to the expressions sec arc tan Y and tan arc sec Y without first having to determine the angle. In addition sec Xg can be obtained with greater accuracy on the hyperbolic scale than from cos Xg on scale $^1/y$.

The **relations between the trigonometric functions** for single, double and halved angles and the algebraic functions derived from them are shown in tabular form on page 29 together with the analogous relations between the hyperbolic functions.

The Basic Exponential Functions e^{-X^g} and e^X and the inverse logarithmic functions $(-\log Y)^g$ and log Y

The basic function e^{-x^g} is represented on the logarithmic slide rule by a uniformly divided scale. It is therefore suitable for any desired extension of the range. The stages of the e^{-x^g} scale belonging to the various decades of the principal scale only differ by whole multiples of $\frac{200}{\pi}$ log 10 as an additive constant.

The "Mathema Rule" (new design) has a scale $e^{x^g} \ldots ^g$ In Y, running to the right.

The details given on pp. 23 and 24 in the Mathema Leaflet, concerning the scale (running in the reverse direction) for the exponential Heaviside function, with the arguments in "new degrees", apply here likewise (mutatis mutandis).

The introduction of the new cursor enables us to dispense with the rules given on pages 24 and 25 for the number of digits in the results, since the cursor gives the power of ten as factors of Y, which belongs to the individual additive constants of X.

The possibility discussed on p. 28 of composing sinh x^g and cosh x^g from the values of 0·5 $e^{\pm x^g}$ using the scale $e^{x^g} \ldots ^g$In Y, can be easily carried out if the slide is left in its basic position. In larger arguments the member 0·5 e^{-x^g} is usually to be left out of account, so that we may say that sinh x^g = cosh x^g = 0·5 e^{x^g}, thus obtaining the continuation of the sinh and cosh scales provided on the back of the slide. For the terms composed with 0·5 e^{x^g} the following applies:

To calculate f(x) ÷ 0·5e^{x^g} or 0·5 e^{x^g} ÷ f (x), y (= 2) is placed underneath the cursor-mark set to e^{x^g}, and the reading is taken, above Y = f (X), on the y scale or the 1/y scale, as the case may be.

Example: sin 260^g ÷ 0·5 e^{260^g} = —0·02725 = —1/36·7.

To calculate 0·5 e^{x^g} f (x) or 1/0·5 e^{x^g} f (x), we first of all place y = e^{x^g} (leaving the position of the cursor for the moment unaltered) above Y = 2, then taking the reading above Y = f (x) on the y scale or the 1/y scale as the case may be.

Example: 0·5 e^{330^g} sin 330^g = —79·4 = —1/0·01259. (The exponent ^{g} means ^{d} or "degree".)

n	$200 \div \pi \times n \ln 10$	n	$200 \div \pi \times n \ln 10$	n	$200 \div \pi \times n \ln 10$
1	146·587 119g	4	586·348 476g	7	1026·109 834g
2	293·174 238g	5	732·935 595g	8	1172·696 953g
3	439·761 357g	6	879·522 715g	9	1319·284 072g

In order to make the additive constant, at least for the first 7 stages, a round figure suitable for mental calculation, namely to 140g, the cursor of the Mathema Slide Rule is provided with marks which have been displaced by a corresponding amount. The edge of the cursor is correspondingly labelled on the top narrow edge.

The number of places of decimals in the result x^{-x^g} within the range of the principal decade of scale Y is determined from the fact that the number of noughts after the decimal point must be equal to the factor of the additive constants. Accordingly, the number of places of decimals of the result of e^{+x^g} within the range of the principal decade of the $^1/y$ scale in the basic position must be equal to the factor of the additive constants.

$e^{0{\cdot}5\pi} = e^{100\,g} = 4{\cdot}81$; M $^{10}_{x}$ (e^{x^g}) Sc 100g / Y Sc $0{\cdot}481 \times 10 = 4{\cdot}81$

$e^{-0{\cdot}5\pi} = e^{-100g} = 0{\cdot}208$; B pos: M $^{10}_{x}$ (e^{x^g}) Sc 100g / $^1/y$ Sc $2{\cdot}08 \times 10^{-1} = 0{\cdot}208$

$e^{3\pi} = e^{600\,g} = 1{\cdot}239 \times 10^4$; M $^{10^1}_{560}$ (e^{x^g}) Sc 40g / Y Sc $0{\cdot}1239 \times 10^5$

$e^{-3\pi} = e^{-600\,g} = 8{,}07 \times 10^{-5}$; B pos: M $^{10^5}_{560}$ (e^{x^g}) Sc 40g / $^1/y$ Sc $8{\cdot}07 \times 10^{-5}$

$e^{10\pi} = e^{2000\,g} = e^{1465{\cdot}87 + 439{\cdot}76 + 94{\cdot}36} = 4{\cdot}4 \times 10^{10+3}$

M $^{10}_{x}$ (e^{x^g}) Sc 94·36 / Y Sc $0{\cdot}44 \times 10 = 4{\cdot}4$

$\frac{\sinh}{\cosh} 225g = (e^{225} \mp e^{-225}) \div 2 = 17{\cdot}135 \mp 0{\cdot}015 = \frac{17{,}12}{17{,}15}$

$^g\tanh 0{\cdot}99996 = -\frac{200}{\pi} \times \ln \sqrt{(1 - 0{\cdot}99996) \div (1 + 0{\cdot}99996)} = -\frac{200}{\pi} \ln \sqrt{\frac{0{,}00004}{2}} = 344{\cdot}5g$

B pos: $\frac{1}{\sqrt{x}}$ Sc 2 / M $^{4{,}4}_{10^1}$ (e^{x^g}) Sc 1·01; $\frac{200}{\pi}(4{\cdot}4 + 1{\cdot}01) = 344{\cdot}5g$

$e^{0{\cdot}1\pi} \times \sin 40g = e^{20} \times \sin 40g = 0{\cdot}805$

M $^{10}_{x}$ (e^{x^g}) Sc 20g / Y Sc 1·37; y Sc 1 / sin Sc 40g // y Sc 1·37 / Y Sc 0·805

$e^{-0.1\pi} \times \sin 40^g = e^{-20^g} \times \sin 40^g = 0.429$

B pos: $M^{10}_{x}(e^{x^g})$ Sc 20^g / $^1/y$ Sc 0·73; y Sc 1 / sin Sc 40^g // y Sc 0·73 / Y Sc 0·429

amp $60^g = 2^g \tan e^{60^g} - 100^g = 2 \times 76{\cdot}35^g - 100^g = 52{\cdot}7^g$

B pos: $M^{10}_{x}(e^{x^g})$ Sc 60^g / Y Sc 2·566; $^1/y$ Sc 2·566 / tan Sc 23·65; $100^g - 23{\cdot}65^g = 76{\cdot}35^g$

(or simple: tanh Sc 60^g / sin Sc $52{\cdot}7^g$)

The scale for the **basic function e^x** differs from the reverse scale for the function e^{-x} in that it increases from left to right in the basic stage and in the higher stages, by the correctness of the number of places of decimals in the preliminary stage and by the argument being expressed in natural numbers.

As the exponential and logarithmic functions are of equal importance, the Mathema Slide Rule is provided with both the function e^x .. log Y and also the function log X .. e^Y. As a result of this it is always possible to carry out the calculations direct. Nevertheless, it must be borne in mind that the reading accuracy differs as between the basic and inverse functions. As the exponent increases, the reading accuracy of the basic exponential function e^x gets greater, whilst that of the inverse exponential function e^Y decreases. The two degrees of reading accuracy reach a balance with exponent 1 and the result e = 2·718281828459 = 1/0·3678794412. The same applies to the reading accuracy in the case of the logarithms.

The **additive constant** for the different stages of the e^x scale is a whole number multiple of log 10 = 1/M = 2·302585093 = 1/0·434294482.

n	n log 10	n	n log 10	n	n log 10
1	2·302585093	4	9·210340372	7	16·118095651
2	4·605170186	5	11·512925465	8	18·420680744
3	6·907755279	6	13·815510558	9	20·723265837

In order to bring the additive constant for several stages to the figure 2·2, the cursor of the Mathema Slide Rule is provided with displaced marks for this purpose. The edge of the cursor on the narrow side is labelled accordingly.

Examples:

ln $\sqrt{1-0.83^2}$ = ln 0·558 = —0·584; B pos: $\sqrt{1-x^2}$ Sc 0·83 / Y Sc 0·558; $^1/_y$ Sc 0·558 / M^{x}_{10} (e^x) Sc —0·584

ln cos 40^g = ln 0·808 = —0·212; B pos: sin Sc 60^g / Y Sc 0·808; $^1/_y$ Sc 0·808 / M^{x}_{10} (e^x) Sc —0·212

ln sec 40^g = ln $\frac{1}{\cos 40^g}$ = ln 1·238 = 0·212; B pos: sin Sc 60^g / $^1/_y$ Sc 1·238; Y Sc 1·238 / M^{x}_{10} (e^x) Sc 0·212

ln sinh 40^g = ln 0·67 = —0·4; B pos: sinh 40^g / Y Sc 0·67; $^1/_y$ Sc 0·67 / M^{x}_{10} (e^x) Sc —0·4

ln $\sqrt{1.04^2-1}$ = ln 0·285 = —1·253; B pos: $\sqrt{x^2-1}$ Sc 1·04 / Y Sc 0·285; $^1/_y$ Sc 0·285 / M^{x}_{10} (e^x) Sc —1·253

ln tan 30^g = ln 0·51 = —0·674; B pos: tan Sc 30^g / Y Sc 0·51; $^1/_y$ Sc 0·51 / M^{x}_{10} (ex) Sc —0·674

ln tan 70^g = ln cot 30^g = ln $\frac{1}{0.51}$ = 0·674; B pos: tan Sc 30^g / $^1/_y$ Sc 1·96; Y Sc 1·96 / M^{x}_{10} (e^x) Sc 0·674

ln cosh 40^g = ln 1·205 = 0·186; B pos: cosh Sc 40^g / M^{x}_{10} (e^x) Sc 0·186

e^2 = 7·39; M^{x}_{10} (e^x) Sc 2 / Y Sc 0·739 × 10

e^{-2} = 0·1353; B pos: M^{x}_{10} (e^x) Sc 2 / $^1/_y$ Sc 1·353 × 10^{-1}

e^4 = $e^{2.2+1.8}$ = 54·6; $M^{2.2}_{10}$ (e^x) Sc 1·8 / Y Sc 0·546 × 10^2

e^{-4} = $e^{-2.2-1.8}$ = 0·0183; B pos: $M^{2.2}_{10}$ (e^x) Sc 1·8 / $^1/_y$ Sc 1·83 × 10^{-2}

e^{11} = $e^{8.8+2.2}$ = 59900; $M^{8.8}_{10}$ (e^x) Sc 1·8 / Y Sc 0·599 × 10^5

e^{-11} = $e^{-8.8-2.2}$ = 0·0000167; B pos: $M^{8.8}_{10}$ (e^x) Sc 1·8 / $^1/_y$ Sc 1·67 × 10^{-5}

e^{100} = $e^{92.103+7.897}$ = 2·69 × 10^{43}; $M^{6.6}_{10}$ (e^x) Sc 1·297 / Y Sc 0·269 × 10^1

ar sinh 0·6 = ln (0·6 + $\sqrt{1+0.6^2}$) = ln (0·6 + 1·166) = ln 1·766 = 0·569

Y Sc 0·6 / $\sqrt{1+Y^2}$ Sc 1·166; Y Sc (1·166 + 0·6) / M^{x}_{10} (e^x) Sc 0·569; (or y Sc 0·6 / sinh Sc 36.25^g = 0·569)

ar tanh 0·6 = ln $\sqrt{(1+0.6) \div (1-0.6)}$ = 0·693

$\sqrt{x}$ Sc 1·6 / $\sqrt{x}$ Sc 0·4 // y Sc 1 / Y Sc 2; Y Sc 2 / M^{x}_{10} (e^x) Sc 0·693 (or Y Sc 0·6 / tanh Sc 44.1^g = 0·693)

The basic logarithmic functions $\pm$ log X and the inverse exponential functions $e^{\pm Y}$

For basic logarithmic functions and inverse exponential functions the Mathema Slide Rule has two three-stage groups of scales, $Y = \pm \log X$ for positive and negative logarithms, $e^{\pm Y}$ for direct and reciprocal powers of e.
With the logarthmic scales we get not only the natural logarithms on scale Y but also logarithms to any desired **base** a. For this purpose, we place one terminal calibration of the slide opposite the figure a on the logarithmic scale, as we must have:

$$^{a}\log a = 1.$$

The logarithms to base a differ from the natural logarithms by a constant factor, the modulus

$$M_a = {}^{1}/\log a.$$

Of particular importance is the modulus of the decadic or Brigg's logarithms M = 0·4343.

Examples: $^{10}\log 2 = \lg 2 = 0{\cdot}301$ $\qquad \lg 10^x = x.$

Bringing a number a to the power of m with the aid of the slide rule can be explained as forming m log a on scale Y and forming the antilog of this on scale e^Y according to the relationship

$$\log a^m = m \log a.$$

Instead of this, however, it is also possible to determine the antilog of $m \times {}^{a}\log a$ starting from scale y. That this is the rule is apparent from the fact that in this case scale Y is not required at all and neither log a nor m log a is read off.

For **evolution,** the above applies but with $^{1}/n = m$.

The following operations are set out in such a way that it is not necessary to insert the slide.

$-\ln X$	$^{1}/a$	$^{1}/a^{m}$	e^{-Y}
$^{1}/y$	m	1	$^{1}/y$
y	$^{1}/m$	1	y
$\ln X$	a	a^{m}	e^{Y}

$-\ln X$	$^{1}/a$	$^{1}/a^{1/n}$	e^{-Y}
$^{1}/y$	$^{1}/n$	1	$^{1}/y$
y	n	1	y
$\ln X$	a	$a^{1/n}$	e^{Y}

Examples:

$1.05^{20} = 2.653$; 10 ln Sc 1·05 / y Sc 1 // y Sc 2 / 0·1 ln Sc 2·653

$0.95^{20} = 0.3585$; -10 ln Sc 0·95 / y Sc 1 // y Sc 2 / -0·1 ln Sc 0·3585

$\sqrt[5]{0.5} = 0.8705$; —0·1 ln Sc 0·5 / y Sc 5 // y Sc 1 / —ln Sc 0·8705

$2345^{6.7} = 10^{6.7 \times \lg 2345} = 10^{6.7 \times 3.37} = 10^{22 + 0.58} = 3.8 \times 10^{22}$

y Sc 0·1 / 0·1 ln Sc 10 // 0·1 ln Sc 2345 / y Sc 3·37;

6·7 × 3·37 = 22·58; y Sc 1 / 0·1 ln Sc 10 // y Sc 0·58 / 0·1 ln Sc 3·8

$\ln (246 \times 10^{8}) = \ln 246 + 8 \ln 10 = 5.51 + 18.42 = 23.93$

0·1 ln Sc 246 / Y Sc 5·51; 0·1 ln Sc 10 / y Sc 1 // y Sc 8 / Y Sc 18·42

$e^{15} = (e^{7.5})^2 = 1810^2 = 3.27 \times 10^{6}$; Y Sc 7·5 / 0·1 ln Sc 1810;

Y Sc 1810 / $\sqrt{x}$ Sc 3.27×10^{6}

$e^{\sin 40^g} = 1.8$; sin Sc 40g / ln Sc 1·8

$(\ln 10)^2 = 5.3$; 0·1 ln Sc 10 / $\sqrt{x}$ Sc 5·3

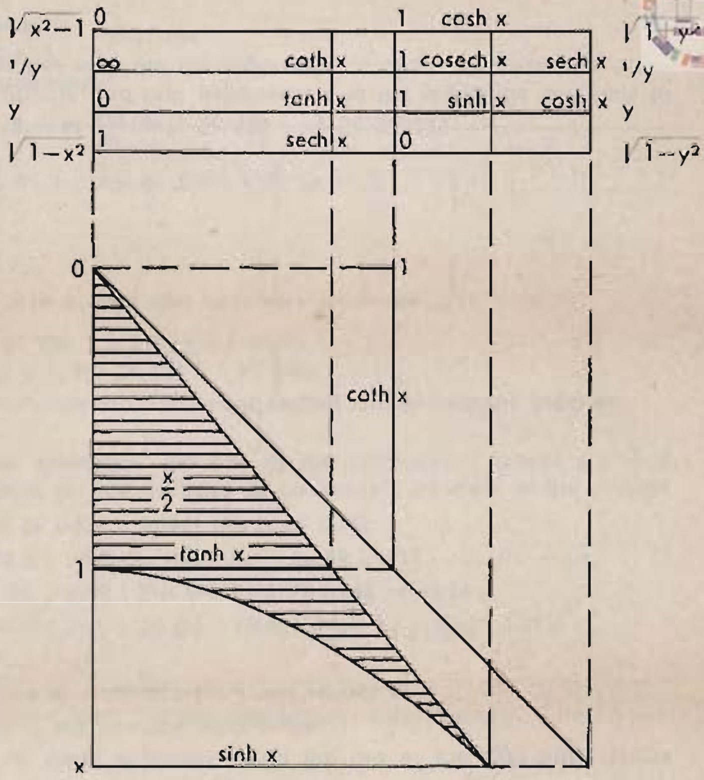
$\ln (\ln 10) = 0.835$; 0·1 ln Sc 10 / $M\frac{x}{10}(e^x)$ Sc 0·835

The two groups of the logarithmic scales show numbers which are **reciprocals** of one another. The values given for $e > X > 1/e$ are more accurately read off than on scale y and scale 1/y and this is more so the nearer the numbers are to 1.

Exaple: $1/1.01234 = 0.98781$.

The Hyperbolic Functions and their inversions

The scales of the **hyperbolic functions** are given on the back of the slide. As in the case of the other scales on the slide and as the variables are again indicated by x and y, the numbers refer to scale y. Passing between the hyperbolic scales and scale y is carried out on the one hand by means of the hair lines on the windows and on the other hand by the terminal calibration of scale Y.



Hyperbolic functions.
The scales of this diagram are not logarithmic.

REFER p-9

Multiply the argument (in radians) by 63·662 to get grades (shown as sinh x^g) and read answer on y. etc.

Examples: sinh 0·6 $\pi/2$ = sinh 60^g = 1·088; 0·1 sinh Sc 60^g / y Sc 1·088
tanh 0·4 $\pi/2$ = tanh 40^g = 0·557; tanh Sc 40^g / Y Sc 0·557
^{g}sinh 0·2 = $12{\cdot}65^g$; y Sc 0·2 / sinh Sc $12{\cdot}65^g$

cosh 0·5 $\frac{\pi}{2}$ = cosh 50^g = 1·325; 0·1 cosh Sc 50^g / y Sc 1·325

$\frac{200}{\pi}$ ar sinh 2 = ^{g}sinh 2 = $91{\cdot}9^g$; y Sc 2 / 0·1 sinh Sc $91{\cdot}9^g$

ln cosh 50^g = ln 1·325 = 0·281; B pos: 0·1 cosh Sc 50^g / M $\frac{x}{10}$ (e^x) Sc 0·281

As cosh x = $\sqrt{1 + \sinh^2 x}$, we can find the values of cosh x for small arguments with the aid of the sinh basic stage scale and the hyperbolic scale $\sqrt{1 + Y^2}$ more accurately than if we use the cosh scale.

In a similar way, sech x = $\sqrt{1 - \tanh^2 x}$ is obtained with greater accuracy for small values of x if we use the circular scale $\sqrt{1 - Y^2}$.

Examples: cosh 10^g = $\sqrt{1 + \sinh^2 x}$ = 1·0123; 10^g = 0·1571; cursor left Y Sc 0·1 / cursor right $\sqrt{1 + Y^2}$ Sc 1·0123
^{g}cosh 1·0246 = 0·223 = $14{\cdot}2^g$; cursor right $\sqrt{x^2 - 1}$ Sc 1·0246 / left cursor Y Sc 0·142 = $14{\cdot}2^g$
sech 10^g = $\sqrt{1 - \tanh^2 x}$ = 0·9876; cursor left Y Sc 0·1 / cursor right $\sqrt{1 - Y^2}$ Sc 0·9876
^{g}sech 0·975 = 0·2225 = $14{\cdot}3^g$; cursor right $\sqrt{1 - x^2}$ Sc 0·975 / cursor left Y Sc 0·143

Accordingly, the relationship cosh x = $1/$sech x could be used for the purpose of converting numbers in the vicinity of 1 into reciprocal numbers. Nevertheless, in spite of greater simplicity, the use of the logarithmic scales ± log X gives more accurate results.

When calculating compound expressions with hyperbolic functions we start with **setting the hyperbolic function.**

Examples: sin 60^g × sinh 60^g = 0·881; 0·1 sinh Sc 60^g / Y Sc 1 // sin Sc 60^g / y Sc 0·881
sin 60^g ÷ sinh 60^g = 0·743; sin Sc 60^g / 0·1 sinh Sc 60^g // y Sc 0·1 / Y Sc 0·743

For values x > 224^d the factor $\frac{e^{-x}}{2}$ can be neglected in the case of sinh and cosh. The following then applies:

sinh x ≈ cosh x ≈ $\frac{e^x}{2}$.

sin 330^g × sinh 330^g = — 79·4; sin 330^g = — sin 70^g

M $\frac{10^4}{280}$(e^{x^g}) Sc 50^g / y Sc 2 // y Sc 1 / Y Sc 89·2; y Sc 89·2 / Y Sc 1 // sin Sc 70^g / y Sc 79·4

sin 260^g ÷ sinh 260^g = — 0·02725; sin 260^g = — sin 60^g

y Sc 1 / sin Sc 60^g // y Sc 2 / Y Sc 1·618; y Sc 0·1 / Y Sc 1·618 // M $\frac{10^3}{140}$ (e^{x^g}) Sc 120 / $1/$y Sc 0·02725

The relations between the hyperbolic functions with single, double and half arguments and the algebraic functions to be obtained from these are shown below in table form together with the corresponding relations between the trigonometric functions.

Relations between the $\frac{\text{trigonometric}}{\text{hyperbolic}}$ functions with single, double and half arguments

sin $\frac{\varphi}{2}$ sin h $\frac{\varphi}{2}$	z	$\sqrt{\pm 1 \mp z^2}$	$1/\sqrt{1/z^2 - 1}$	$\sqrt{\frac{\pm 1 \mp \sqrt{1 \mp z^2}}{2}}$	$\sqrt{\frac{\pm 1 \mp z}{2}}$	$\sqrt{\frac{\pm 1 \mp 1/\sqrt{1 \pm z^2}}{2}}$
cos $\frac{\varphi}{2}$ cos h $\frac{\varphi}{2}$	$\sqrt{1 \mp z^2}$	z	$1/\sqrt{1 - z^2}$	$\sqrt{\frac{1 + \sqrt{1 \mp z^2}}{2}}$	$\sqrt{\frac{1 + z}{2}}$	$\sqrt{\frac{1 + 1/\sqrt{1 \pm z^2}}{2}}$
tan $\frac{\varphi}{2}$ tan h $\frac{\varphi}{2}$	$1/\sqrt{1/z^2 \mp 1}$	$\sqrt{\pm 1/z^2 \mp 1}$	z	$\frac{z}{1 + \sqrt{1 \mp z^2}} = \frac{\pm 1 \mp \sqrt{1 \mp z^2}}{z}$	$\sqrt{\frac{\pm 1 \mp z}{1 + z}}$	$\frac{\mp 1 \pm \sqrt{1 \pm z^2}}{z} = \frac{z}{1 + \sqrt{1 \pm z^2}}$
sin $2\frac{\varphi}{2}$ sin h $2\frac{\varphi}{2}$	$2z\sqrt{1 \mp z^2}$	$2z\sqrt{\pm 1 \mp z^2}$	$\frac{2z}{1 \pm z^2}$	z	$\sqrt{\pm 1 \mp z^2}$	$1/\sqrt{1/z^2 - 1}$
cos $2\frac{\varphi}{2}$ cos h $2\frac{\varphi}{2}$	$1 \mp 2z^2$	$2z^2 - 1$	$\frac{1 \mp z^2}{1 \pm z^2}$	$\sqrt{1 \mp z^2}$	z	$1/\sqrt{1 - z^2}$
tan $2\frac{\varphi}{2}$ tan h $2\frac{\varphi}{2}$	$\frac{2z\sqrt{1 \mp z^2}}{1 \mp 2z^2}$	$\frac{2z\sqrt{\pm 1 \mp z^2}}{2z^2 - 1}$	$\frac{2z}{1 \mp z^2}$	$1/\sqrt{1/z^2 \mp 1}$	$\sqrt{\pm 1/z^2 \mp 1}$	z

The top signs in the above formulae belong to the trigonometric functions and the bottom signs belong to the hyperbolic functions.

The given functions of z for single, double and half arguments are obtained if the value of the initial function for the simple argument of the same column is equal to z. z can originate from a compound expression like a/b or $\sqrt{a/b}$.

The relation of a function with a single, double or half argument to the initial function in the same column with a single argument is obtained by replacing z by the initial function. The expressions can be simplified in a very easy manner if stress is only laid on the double or half argument as the initial value, and not on certain initial functions of these arguments.

An important example of the use of the above relationship is the **goniometric solution of quadratic equations.**

1. $x^2 \pm 2ax - b^2 = 0$.

   Set $b/a = \tan 2\varphi$ and find $x_1 = \pm b \tan \varphi$, $x_2 = \mp b \cot \varphi$.

2. $x^2 \pm 2ax + b^2 = 0$, when $b < a$.

   Set $b/a = \sin 2\varphi$ and find $x_1 = \mp b \tan \varphi$, $x_2 = \mp b \cot \varphi$.

Examples: $x^2 + 234\,x - 5670 = 0$; $a = 117$; $b = 75{\cdot}2$; $x_1 = 22{\cdot}1$; $x_2 = -256{\cdot}1$

Y Sc 75·2 / y Sc 1·17 // y Sc 1 / tan Sc 36·36g; φ = 18·18g

Y Sc 1 / y 75·2 // tan Sc 18·18g / y Sc 22·1; y Sc 1 / Y Sc 75·2 // tan Sc 18·18g / 1/y Sc 256·1

$x^2 + 234 + 5670 = 0$; $a = 117$; $b = 75{\cdot}2$; $x_1 = -27{\cdot}4$; $x_2 = -206{\cdot}5$

Y Sc 75·2 / y Sc 1·17 // y Sc 1 / sin Sc 44·49; φ = 22·29

y Sc 75·2 / Y Sc 1 // tan Sc 22·29 / y Sc 27·4; y Sc 1 / Y Sc 75·2 // tan Sc 22·29 / 1/y Sc 206·5

Differential quotients and indeterminate integrals of the elementary functions of the "Mathema" Slide Rule

$f'(x)$	$f(x)$	$\int f(x)\,dx$
mx^{m-1}	x^m	$x^{m+1}/(m+1) + C$
$-x/\sqrt{1-x^2}$	$\sqrt{1-x^2}$	$0{,}5\,x\sqrt{1-x^2} + 0{,}5 \arcsin x + C$
$x/\sqrt{1+x^2}$	$\sqrt{1+x^2}$	$0{,}5\,x\sqrt{1+x^2} + 0{,}5 \operatorname{arcsinh} x + C$
$x/\sqrt{x^2-1}$	$\sqrt{x^2-1}$	$0{,}5\,x\sqrt{x^2-1} - 0{,}5 \operatorname{arccosh} x + C$
e^x	e^x	$e^x + C$
$1/x$	$\ln x$	$x \ln x - x + C$
$\cos x$	$\sin x$	$-\cos x + C$
$-\sin x$	$\cos x$	$\sin x + C$
$\sec^2 x$	$\operatorname{tg} x$	$-\ln \cos x + C$
$-\operatorname{cosec}^2 x$	$\operatorname{cotg} x$	$\ln \sin x + C$
$\sin x/\cos^2 x$	$\sec x$	$\ln \sqrt{\frac{1+\sin x}{1-\sin x}} + C = \ln \operatorname{tg}(\pi/4 + x/2) + C = \operatorname{ar\,amp} x + C$
$-\cos x/\sin^2 x$	$\operatorname{cosec} x$	$-\ln \sqrt{\frac{1+\cos x}{1-\cos x}} + C$
$1/\sqrt{1-x^2}$	$\arcsin x$	$x \arcsin x + \sqrt{1-x^2} + C$
$-1/\sqrt{1-x^2}$	$\arccos x$	$x \arccos x - \sqrt{1-x^2} + C$
$1/(1+x^2)$	$\operatorname{arctg} x$	$x \operatorname{arctg} x - \ln \sqrt{1+x^2} + C$
$-1/(1+x^2)$	$\operatorname{arccotg} x$	$x \operatorname{arccotg} x + \ln \sqrt{1+x^2} + C$
$\cosh x$	$\sinh x$	$\cosh x + C$
$\sinh x$	$\cosh x$	$\sinh x + C$
$\operatorname{sech}^2 x$	$\operatorname{tgh} x$	$\ln \cosh x + C$
$-\operatorname{cosech}^2 x$	$\operatorname{cotgh} x$	$\ln \cosh x + C$
$-\sinh x/\cosh^2 x$	$\operatorname{sech} x$	$2 \operatorname{arctg} e^x + C = \operatorname{amp} x + \pi/2 + C$
$-\cosh x/\sinh^2 x$	$\operatorname{cosech} x$	$-\ln \sqrt{\frac{\cosh x + 1}{\cosh x + 1}} + C$

$f'(x)$	$f(x)$	$\int f(x)\,dx$
$1/\sqrt{x^2+1}$	$\operatorname{arsinh} x$	$x \operatorname{arsinh} x - \sqrt{x^2+1} + C$
$1/\sqrt{x^2-1}$	$\operatorname{arcosh} x$	$x \operatorname{arcosh} x - \sqrt{x^2-1} + C$
$1/(1-x^2)$	$\operatorname{arctgh} x$	$x \operatorname{arctgh} x + \ln \sqrt{1-x^2} + C$
$1/(1-x^2)$	$\operatorname{arccotgh} x$	$x \operatorname{arccotgh} x + \ln \sqrt{x^2-1} + C$

Power series of the real and complex elementary functions ($z = x + iy$)

$$\sqrt{1+z} = \sum_{m=0}^{\infty} \binom{0,5}{m} z^m = 1 + z/2 - z^2/8 + z^3/16 - z^4/25,6 + z^5/36,57143 - z^6/48,76190 + \,.\,. \qquad |z| < 1$$

$$\sqrt{z \pm 1} = \sqrt{z} \sum_{m=0}^{\infty} (-1)^m \binom{0,5}{m} /z^m = \sqrt{z}\,(1 \pm 1/2z - 1/8z^2 \pm 1/16z^3 - 1/25,6z^4 \pm 1/36,57143z^5 - \,.\,.) \qquad |z| > 1$$

$$e^z = \sum_{n=0}^{\infty} z^n/n! = 1 + z + z^2/2 + z^3/6 + z^4/24 + z^5/120 + z^6/720 + z^7/5040 + z^8/40320 + \,.\,. \qquad |z| < \infty$$

$$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^n/n = z - z^2/2 + z^3/3 - z^4/4 + z^5/5 - z^6/6 + z^7/7 - z^8/8 + z^9/9 - z^{10}/10 + \,.\,. \qquad |z| < 1$$

$$\begin{matrix} \sin z \\ \sinh z \end{matrix} = \sum_{n=0}^{\infty} (\mp 1)^n z^{2n+1}/(2n+1)! = z \mp z^3/6 + z^5/120 \mp z^7/5040 + z^9/362880 \mp z^{11}/39\,916880 + \,.\,. \qquad |z| < \infty$$

$$\begin{matrix} \cos z \\ \cosh z \end{matrix} = \sum_{n=0}^{\infty} (\mp 1)^n z^{2n}/(2n)! = 1 \mp z^2/2 + z^4/24 \mp z^6/720 + z^8/40320 \mp z^{10}/3\,628800 + z^{12}/\,479\,001\,600 + \,.\,. \qquad |z| < \infty$$

$$\begin{matrix} \mathrm{tg}\ z \\ \mathrm{tgh}\ z \end{matrix} = \sum_{n=1}^{\infty} (+1)^{n-1} 2^{2n}(2^{2n}-1) B_{2n} z^{2n-1}/(2n)! = z \pm z^3/3 + z^5/7,5 \pm z^7/18,52941 + z^9/45,72581 \pm z^{11}/112,82562 + \,.\,. \qquad |z| < \pi/2$$

$$\begin{matrix} \mathrm{cotg}\ z \\ \mathrm{cotgh}\ z \end{matrix} = \sum_{n=0}^{\infty} (\mp 1)^n 2^{2n} B_{2n} z^{2n-1}/(2n)! = 1/z + z/3 - z^3/45 + z^5/472,5 - z^7/4725 + z^9/46777,5 - z^{11}/462020,9 + \,.\,. \qquad |z| < \pi$$

$$\left.\begin{array}{l}\sec z\\ \operatorname{sech} z\end{array}\right\} = \sum_{n=0}^{\infty} (\mp 1)^n E_{2n}\, z^{2n}/(2n)! = 1 \pm z^2/2 + z^4/4{,}8 \pm z^6/11{,}80328 + z^8/29{,}11191 \pm z^{10}/71{,}82756 + \ldots \qquad |z| < \pi/2$$

$$\left.\begin{array}{l}\operatorname{cosec} z\\ \operatorname{cosech} z\end{array}\right\} = \sum_{n=0}^{\infty} (\mp 1)^n (2^{2n}-2)\, B_{2n}\, z^{2n-1}/(2n)! = 1/z \pm z/6 + z^3/51{,}42857 \pm z^5/487{,}74193 + z^7/4762{,}20472 \pm \ldots \qquad |z| < \pi$$

$$\left.\begin{array}{l}\arcsin z\\ \operatorname{arsinh} z\end{array}\right\} = z + \sum_{n=1}^{\infty} (\pm 1)^n (2n-1)!!\, z^{2n+1}/(2n+1)\,(2n)!! = z \pm z^3/6 + z^5/13{,}33333 \pm z^7/22{,}4 + z^9/32{,}91429 \pm z^{11}/44{,}69841 + \ldots \qquad |z| < 1$$

$$\arccos z = \pi/2 - \arcsin z$$

$$\operatorname{arcosh} z = \ln 2z - \sum_{n=1}^{\infty} (2n-1)!!/(n \cdot 2^{2n+1} n!\, z^{2n}) = \ln 2z - 1/4z^2 - 1/10{,}66667z^4 - 1/19{,}2z^6 - 1/29{,}25714z^8 - \ldots \qquad |z| > 1$$

$$\left.\begin{array}{l}\operatorname{arctg} z\\ \operatorname{arctgh} z\end{array}\right\} = \sum_{n=0}^{\infty} (\mp 1)^n z^{2n+1}/(2n+1) = z \mp z^3/3 + z^5/5 \mp z^7/7 + z^9/9 \mp z^{11}/11 + z^{13}/13 \mp z^{15}/15 \ldots \qquad |z| < 1$$

$\operatorname{arccotg} z = \operatorname{arctg} {}^1/z$
$\operatorname{arccotgh} z = \operatorname{arctgh} {}^1/z$

$\operatorname{arcsec} z = \arccos {}^1/z$
$\operatorname{arsech} z = \operatorname{arcosh} {}^1/z$

$\operatorname{arccosec} z = \arcsin {}^1/z$
$\operatorname{arcosech} z = \operatorname{arsinh} {}^1/z$

$(2n)!! = 2 \cdot 4 \cdot 6 \ldots 2n$
$(2n-1)!! = 1 \cdot 3 \cdot 5 \ldots (2n-1)$

The limitation to the first terms of the power series results in approximation formulae for the functions in the event of small or large arguments.

Complex functions, conformal representations and plane orthogonal systems of co-ordinates

$z = x + iy$

$r = |z| = \sqrt{x^2 + y^2}$

$\varphi = \arg z = \operatorname{arctg} y/x \pm 2\pi m$

$z = r(\cos\varphi + i\sin\varphi) = r\,e^{i\varphi} = r\,i^{2\varphi/\pi}$ (Euler)

$e^z = e^x(\cos y + i\sin y)$

$\cos\varphi = (e^{i\varphi} + e^{-i\varphi})/2$

$\sin\varphi = (e^{i\varphi} - e^{-i\varphi})/2i$

$\sin ix = i\sinh x$

$\cos ix = \cosh x$

$\operatorname{tg} ix = i\operatorname{tgh} x$

$\operatorname{cotg} ix = -i\operatorname{cotgh} x$

$\sin z = \sin x\cosh y + i\cos x\sinh y$

$\cos z = \cos x\cosh y + -i\sin x\sinh y$

$$\operatorname{tg} z = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\operatorname{cotg} z = \frac{\sin 2x - i\sinh 2y}{-\cos 2x + \cosh 2y}$$

$x = |z|\cos\varphi$

$y = |z|\sin\varphi$

m = nombre entier

$z^n = r^n(\cos n\varphi + i\sin n\varphi) = r^n e^{in\varphi} = r^n\,i^{2n\varphi/\pi}$ (Moivre)

$\ln z = \ln|z| + i\varphi$

$\operatorname{Ln} z = (\ln z)_{m=0}$

$\sinh ix = i\sin x$

$\cosh ix = \cos x$

$\operatorname{tgh} ix = i\operatorname{tg} x$

$\operatorname{cotgh} ix = -i\operatorname{cotg} x$

$\sinh z = \sinh x\cos y + i\cosh x\sin y$

$\cosh z = \cosh x\cos y + i\sinh x\sin y$

$$\operatorname{tgh} z = \frac{\sinh 2x + i\sin 2y}{\cosh 2x + \cos 2y}$$

$$\operatorname{cotgh} z = \frac{\sinh 2x - i\sin 2y}{\cosh 2x - \cos 2y}$$

$\arcsin iz = i \operatorname{arsinh} z = i \ln (z + \sqrt{1+z^2})$

$\arccos iz = -i \operatorname{arcosh} iz = \pm i \ln (z + \sqrt{1+z^2}) + \pi/2$

$\operatorname{arctg} iz = i \operatorname{arctgh} z = i \ln \sqrt{\frac{1+z}{1-z}}$

$\operatorname{arccotg} iz = -i \operatorname{arccotgh} z = -i \ln \sqrt{\frac{z+1}{z-1}}$

$\arcsin z = -i \operatorname{arsinh} iz = -i \ln (iz + \sqrt{1-z^2})$

$\arccos z = -i \operatorname{arcosh} z = \pm i \ln (z + i \sqrt{1-z^2})$

$\operatorname{arctg} z = -i \operatorname{arctgh} iz = -i \ln \sqrt{\frac{1+iz}{1-iz}}$

$\operatorname{arccotg} z = i \operatorname{arccotgh} iz = -i \ln \sqrt{\frac{iz-1}{iz+1}}$

For an **analytical function** $Z = f(z) = X(x, y) + iY(x, y)$, the **Cauchy-Riemann** differential equation

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y} \qquad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}$$

and the **Laplace** potential equation

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 X}{\partial y^2} \qquad \frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} = 0$$

apply.

An analytical function $Z = f(z)$ provides, for a curve of the z plane, a **conformation representation** in the Z plane, and vice versa.

$X(x, y)$ and $Y(x, y)$ are harmonic functions.

The conformal representation of an orthogonal system of co-ordinates again results in an orthogonal system of co-ordinates.

Parabolic co-ordinates

$Z = z^2$ $\qquad z = \sqrt{Z}$

The Z plane has two **Riemann surfaces** with the **ramification points** $Z = 0$ and $Z = \infty$; at these **singular points,** $f'(Z) = 0$ or $= \infty$.

$X + iY = x^2 - y^2 + 2ixy$

$Y_1 = 2y\sqrt{y^2 + X}$, confocal parabolas for $y = c$.

$Y_2 = 2x\sqrt{x^2 - X}$, confocal parabolas for $x = c$.

$Re^{i\varphi} = r^2 e^{2i\varphi}$, polar co-ordinates with duplicated arguments.

$x + iy = \sqrt{(R + X)/2} + i\sqrt{(R - X)/2}$

$y_1 = Y/2x$, equilateral hyperbolas for $Y = c$.

$y_2 = \sqrt{x^2 - X}$, equilateral hyperbolas for $X = c$.

$re^{i\varphi} = \sqrt{R}\, e^{i\varphi/2}$, polar co-ordinates with "halved argument".

Polar co-ordinates

$Z = e^z$ $\qquad$ $z = \ln Z$

To the 4 parallel strips of the width $\pi/2$ for $y = 0$ to $y = 2\pi$ of the z plane correspond the 4 quadrants of the Z plane, which has an infinite number of Riemann surfaces.

$X + iY = e^x \cos y + i\, e^x \sin y$ $\qquad$ $x + iy = \ln R + i\varphi$

$Y_1 = X \tan y$, straight lines through the zero point for $y = c$. $\qquad$ $y_1 = \arcsin Ye^{-X} \pm 2\pi m$

$Y_2 = \sqrt{e^{2x} - X}$, circles about the zero point for $x = c$. $\qquad$ $y_2 = \arccos Xe^{-x} \pm 2\pi m$

$Re^{i\psi} = e^x \times e^{iy}$ $\qquad$ $r(\cos\varphi + i \sin\varphi) = \ln R + i\psi$

$R = e^x$ $\qquad$ $r = \sqrt{(\ln R)^2 + \psi^2}$

$= y$ $\qquad$ $\varphi = \arccos((\ln R)/r)$

Transformation by reciprocal radii, inversion

$Z = 1/z$ $\qquad$ $z = 1/Z$

$X + iY = x/r^2 - iy/r^2$

$Y_1 = -1/2y + \sqrt{1/4y^2 - X^2}$, circular clusters for $y = c$, contacting the real axis: dipol.

$Y_2 = +\sqrt{X/x - X^2}$, circular clusters for $x = c$, contacting the imaginary; dipol.

$Re^{i\psi} = 1/re^{i\varphi}$

$R = 1/r$, reflection with respect to circle of unit radius. $\qquad$ $\psi = -\varphi$, reflection with respect to the real axis.

Elliptical co-ordinates

Confocal ellipses and confocal hyperbolas with the focal points in $Z = +1$ or $Z = \pm i$.

$Z = \sin z$;	$X^2/\cosh^2 y + Y^2/\sinh^2 y = 1$;	$X^2/\sin^2 x - Y^2/\cos^2 x = 1$.
$Z = \cos z$;	$X^2/\cosh^2 y + Y^2/\sinh^2 y = 1$;	$X^2/\cos^2 x - Y^2/\sin^2 x = 1$.
$Z = \sinh z$;	$-X^2/\cos^2 y + Y^2/\sin^2 y = 1$;	$X^2/\sinh^2 x + Y^2/\cosh^2 x = 1$.
$Z = \cosh z$;	$X^2/\cos^2 y - Y^2/\sin^2 y = 1$;	$X^2/\cosh^2 x + Y^2/\sinh^2 x = 1$.

The image of the co-ordinates for $Z = \sin z$, $\cos z$ and $\cosh z$ agree with one another (focal points at $Z = +1$), while those of the co-ordinates for $Z = \sinh z$ are rotated by 1 ∟ in respect of the foregoing.

Superimposition of the Cartesian and the reciprocal network; dipole in straight flow

$Z = z + 1/z$; $iZ = iz - 1/iz$ $\qquad$ $z = 0{\cdot}5\, Z \pm \sqrt{0{\cdot}25\, Z^2 - 1}$

$X + iY = x + x/r^2 + i\,(y - y/r^2)$

$= (r + 1/r) \cos\varphi + i\,(r - 1/r) \sin\varphi$

The Cartesian co-ordinates of the z plane provide in the Z plane the image of the flow about the circle of unit radius (flow lines and equipotential lines), and likewise that of the flow in the said circle.
The unit circle of the z plane tends towards the straight line from $X = -2$ to $X = +2$ of the Z plane.
The other circles about the zero point of the z plane provide confocal ellipses in the Z plane with the focal points in $X = \pm 2$, while the straight lines through the zero point of the z plane become confocal hyperbolas in the Z plane with the aforementioned focal points.
Circles through the points $z = \pm 1$ tend towards circular arcs of the Z plane.
Circles through the point $z = -1$ with the point $z = +1$ in its interior tend towards Zhukovsky airfoil profiles.

Source and negative source of the same thickness, bipolar co-ordinates

$Z = \tanh z/2$ $\qquad z = \ln(Z+1) - \ln(Z-1) = \ln \dfrac{Z+1}{Z-1}$

$Z = \coth z/2$ $\qquad z = \ln(1+Z) - \ln(1-Z) = \ln \dfrac{1+Z}{1-Z}$

$R_1 = |\operatorname{cosec} y|$, circular clusters for $y = c$ about $X = 0$, $Y = \pm \cot y$ through the poles $X = \pm 1$.
$R_2 = |\operatorname{cosech} x|$, Apollonic circular clusters for $x = c$ about $X = \pm \coth x$, $Y = 0$.

$Z = \tan z/2$ $\qquad z = -i \ln(1+iZ) + i \ln(1-iZ) = -i \ln \dfrac{1+iZ}{1-iZ}$

$Z = \cot z/2$ $\qquad z = -i \ln(iZ-1) + i \ln(iZ+1) = i \ln \dfrac{iZ+1}{iZ-1}$

$R_1 = |\operatorname{cosech} y|$, Apollonic circular clusters for $y = c$ about $x = 0$, $Y = \pm \coth y$.
$R_2 = |\operatorname{cosec} x|$, circular clusters for $x = c$ about $X = \pm \cot y$, $Y = 0$ through the poles $Y = \pm 1$.

Two sources or negative sources of the same thickness

$Z = \sqrt{e^z + 1}$

$$z = \ln(Z-1) + \ln(Z+1) = \ln(Z^2 - 1)$$
$$x = 0{\cdot}5 \ln(R^4 - 2R^2 \cos 2\Psi + 1)$$
$$y = \arctan \frac{\sin 2\Psi}{\cos 2\Psi - 1/R^2}$$

The parallels to the real axis in the z plane become hyperbolas in the Z plane; they pass through the poles $X = \pm 1$, their asymptotes through the zero point.
The parallels to the imaginary axis in the z plane become confocal Cassini curves in the Z plane with the focal points in $X = \pm 1$; in particular, the imaginary axis of the z plane becomes the lemniscate of the Z plane, in accordance with the equation $R = \sqrt{2 \cos 2\Psi}$.

CONVERSION TABLE

in : mm	::	5 : 127
ft : m	::	292 : 89
yds : m	::	35 : 32
miles : km	::	87 : 140
in^2 : cm^2	::	31 : 200
ft^2 : m^2	::	140 : 13
yds^2 : m^2	::	61 : 51
in^3 : cm^3	::	36 : 590
ft^3 : m^3	::	106 : 3
ft^3 : l	::	3 : 85
lbs : kg	::	280 : 127

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