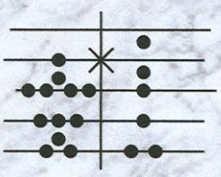
The Fascinating Development of ARITHMETIC,

MCMXXXVII



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From its Origin to the 20th Century and the Methods and Aids used

By

J.H. Donners
Eindhoven
2003

$^{10}\log 409{,}261 = 5.612$

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This book is a slightly extended translation of:

Mijnheer Van Dalen Krijgt Antwoord
(Mr. Van Dalen gets answer)

The Dutch title refers to a memory aid used at school for remembering the priority rules of calculation operators, the initial letters of the words:

Mijnheer Van Dalen Wacht Op Antwoord which translates to:
Mr. Van Dalen is waiting for answer, the dutch equivalent of
My Dear Aunt Sally – multiplication/division/addition/subtraction

M	Machtsverheffen	Raise to a power,
V	Vermenigvuldigen	Multiplication,
D	Delen	Division,
W	Worteltrekken	Extracting roots,
O	Optellen	Addition,
A	Aftrekken	Subtraction.

giving the order of execution of the six species or arithmetical operations.

I would like to thank my son Maurice for giving me advice and for polishing my English. However, should any errors have slipped in, then these would be entirely my responsibility.

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Preface

In the early seventies of the last century, the elektronic calculator made a start. In very short time all calculators – based on logarithms – used to that time were replaced. Slide rules, discs, and cilindrical rules disappeared in drawers or on attics, or even worse, were thrown away.

Some time ago I started colecting those calculators based on logarithms. I made a start with my own wooden Nestler-slide rule with which I worked during about 20 year. Now I have slide rules, discs and slide cards in many sizes, of various materials, for all kinds of applications.

I got interested in who invented the slide rule and the logarithm, also when and why. In that search it was striking that mathematics was practiced already very long on a rather high level; think about Pythagoras, who died ca. 500 B.C. and the theorem named after him, and many other Greeks, whereas on the other hand common calculation i.e. multiplication and division in the 17th century stil was experienced as something very difficult (See Samuel Pepys, chapter 16). Especially the latter interested me: how were calculations performed at that time? No book could give me a survey of the development of the common calculating from the origins to the recently used method. Therefore I tried to collect those data from encyclopedia, dozens of books and the internet and after that to present it organized in this work.

Astronomers, as Nicolaus Copernicus, Tycho Brahe and Johannes Kepler had to perform many complicated calculations. After the publication of the first logarithmic tabel, among others, Johannes Kepler was very interested. All astronomers, from the ancient Greek to Kepler had a great need to simplify calculations. From ancient times the astronomers were the driving force for the development of mathematics and arithmetics. Therefore attention is also paid to the development of astronomy.

To understand the progress of arithmetic is to understand the beginnings of the science of the Egyptians and the Babylonians, the Greeks, the Romans, the Hindus and the Arabs, who all contributed to the development of our modern science and, in particular, of arithmetic.
Besides to logarithms, their development and applications, extensive attention is given to Rabdologia, or calculating with bones or rods. After Genaille-Lucas, a global survey of mechanical calculation and their mechanisms will be given in part one.

In part two, the supplement, attention is paid to tally-stics, finger numbers and finger reckoning, and the calculation methods used by the Chinese, Japanese, the Mayas and the Incas.

Chapter one, *Egypt*

1.1 The Rosetta Stone [1)]

Till the 4th millennium B.C. the ancient Egyptians lived in small communities that clung to the fertile strip of land bordering the Nile. Between 3500 and 3100 B.C., these self-sufficient agricultural communities had gradually coalesced into larger units until there were only two kingdoms: Upper Egypt and Lower Egypt. Then, about 3100 B.C. these regions were united by military conquest from the south by a ruler named Narmer (Menes?). His achievements were, as usual for long times, recorded in hieroglyphics, the earliest form of Egyptian writing, the picture script in which each character represents a concrete object. In this picture script on a macehead of Narmer the number of prisoners and the number of captive animals were recorded, under which 1,422,000 goats. Maybe this number is exaggerated for more splendour, but it points out that very large numbers had made a long development. Hieroglyphic writing is quite suitable for inscribing public monuments or temple walls and was used in this capacity right up into the Christian era.

Officers of Napoleon's army uncovered a polished black basalt stone near the Rosetta branch of the Nile in 1799, when they were digging the foundation of a fort.

The Rosetta Stone, see Figure 1.1, is inscribed in three different type of writing: in the upper third hieroglyphics or 'sacred signs', demotic script in the middle (a later script from 7th century B.C.) and Greek down the bottom third.
These hieroglyphics were deciphered by help of this stone by the Frenchman Jean François Champollion in 1822.

Protected from external invasion by the same deserts that isolated her, Egypt was able to develop the most stable and long lasting of the ancient civilisations.

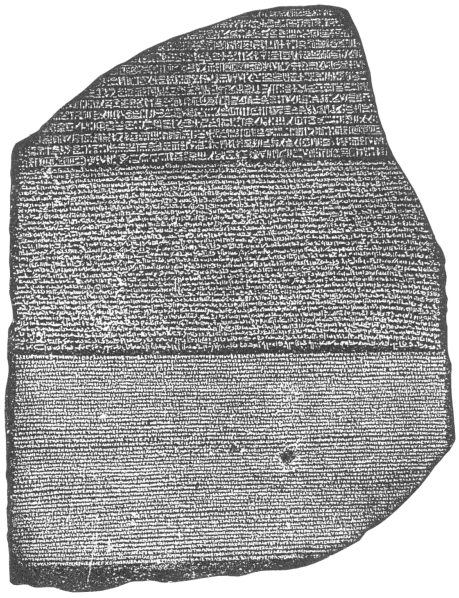


Figure 1.1

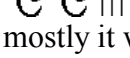
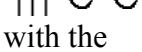
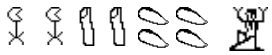
Nevertheless there was trade with neighbouring communities, they conquered adjacent territories and they were also ruled by foreigners. In spite of this there was an uninterrupted, slow development without new ideas. Besides, the Egyptians were extremely conservative which appears from using the name 'the two empires' thousands of years although Upper Egypt and Lower Egypt were united for that long time.
As at many communities all over the world the Egyptians used also the so-called finger reckoning. More about finger reckoning see chapter 16.

There is a statement, made by Herodotus (~ 485 to ~ 425BC) when writing about the Egyptians, that *they write their characters and reckon with pebbles, bringing their hand from right to left while the Greeks go from the left to the right.*
To 'reckon with pebbles', in later time with counters**,** was the fashion to the 17th century and will be discussed in the chapters on the Greek to West Europe from 1000 to 1600 A.D.

First we get acquainted with the hieroglyphic numbers.

1.2 The hieroglyphic numbers

1	10	100	1000
a vertical stroke	a horse heel	a coiled rope	a lotus flower
10,000	100,000	1,000,000	10,000,000
a finger?	a tadpole	the god Heh	rising sun ?

The Egyptian number system is the classical example of an additive system. The value of a number in an additive system has nothing to do with the position of any symbol within the string representing a number: 203 could be written as  or as 
Although the value of the number in both cases is 203, mostly it was written with the smallest number at the left, so  means 1,422,000.

The addition of two numbers was thought of in terms of one number ‘going in’ the second, the symbolic designation is , a pair of walking legs ‘going in’. For subtraction, walking legs ‘going out’ were used: . This raises an interesting question: which direction of motion correspond to ‘in’ and which to ‘out’? The Egyptians did not write from right to left consistently. Their hieroglyphic writing system carries build-in information to give the reader the correct direction. Many of the hieroglyphic symbols are animals or birds, and the reading rule (for text written in a row) is simply: read from head to tail. Feet against the direction of reading are ‘going in’ and denote addition.

Characteristics of the hieroglyphic number system:
it is decimal, it is not positional, there is no zero, (the Egyptians did not need one).

For performing addition, it was necessary only to collect symbols and exchange ten like symbols for the next higher symbol. The same process in reverse performed subtraction.

Egyptian priest developed a more rapid, less pictorial style that was better adapted to pen and ink. In this so-called ‘hieratic’ (sacred) script, the symbols were written in a cursive hand. The Rhind Mathematical Papyrus was written in this script.

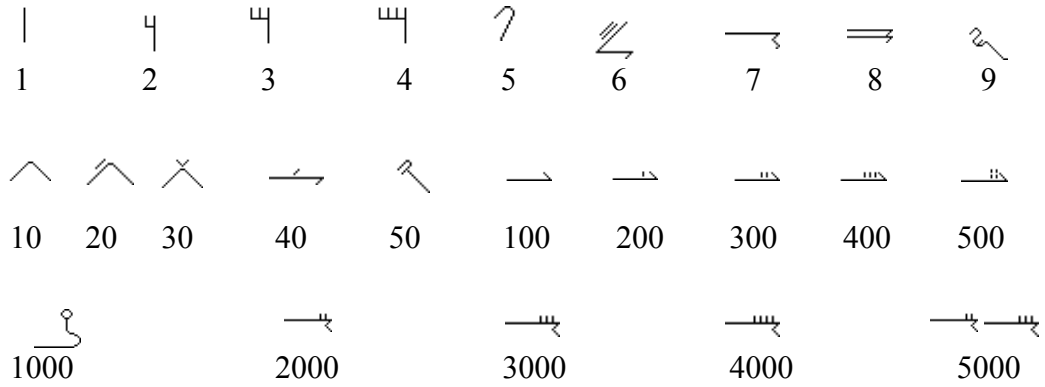
1.3 The Rhind Mathematical Papyrus [2)]

This papyrus was found at Thebes in a room of a ruined building near the Rameseum. In 1858 a Scotsman, the archaeologist, A. Henry Rhind, after whom it is named, purchased it. In 1864 it was willed to the British Museum in two pieces. In 1922 the missing part was found and added. This Rhind Papyrus starts with:

Accurate reckoning. The entrance into the knowledge of all existing things and all obscure secrets.This book was copied in the year 33, in the fourth month of the inundation season, under the majesty of the King of Upper and Lower Egypt, ‘A-user-Re’ (ca 1650 BC) . . . *endowed with life, in likeness to writings of old made in the time of the King of Upper and Lower Egypt, ‘Ne-mat'et-Re’* (1845 – 1801BC). *It is the scribe A'h mes who copies this writing.*

This Rhind papyrus is a practical handbook of mathematical exercises with 85 problems and their solutions and tables. The Rhind Rapyrus obviously was intended for instruction at a school for civil servants, the so-called scribes.

1.4 The hieratic figures / numbers



In hieratic writing form, numerical representation was still additive, based on powers of ten; but the repetitive principle of hieroglyphics was replaced by the device of using a single mark to represent a collection of like symbols. This type of notation may be called cipherization. Like our handwriting is different from individual to individual, there is much difference between written hieratic numbers.

1.5 Multiplication

Multiplication of two numbers was accomplished by successively doubling one of the numbers and than adding the appropriate duplications to form the product. (The technical term for multiplication is "…add, start with..."). To find the product of 12 and 13 for instance, assume the multiplicand to be 13, doubling thus (in Arabic figures):

	1	13
	2	26
/	4	52
/	8	104
totals	12	156

because 12 = 4 + 8, let us put checks (/) alongside these multipliers to indicate that they should be added. Adding those numbers in the right-hand column opposite the checks, the Egyptian mathematician would get the required answer, that is 52 + 104 = 156

With this they discovered the powers of two: 1 2 4 8 16 32 64 128 etc.

1.6 Division

Division was the simple inverse operation to multiplication, where the divisor is repeatedly doubled to give the dividend. Example (Rhind problem nr 69): Multiply 80 (or literally: add, start with 80) till you get 1120. In Arabic figures: 1120 ÷ 80.

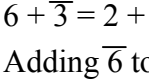
	1	80
/	10	800
	2	160
/	4	320
totals	14	1120

Finding that 800 + 320 = 1120, one adds the powers of two corresponding to the checked numbers, namely 10 + 4 = 14 which gives the disired quotient. The Egyptian division procedure has the pedagogical advantage of not appearing to be a new operation.

The Egyptians required ordinary arithmetic in the daily transactions of commerce and state government to fix taxes, to compute wages and construct a workable calendar. Simple geometric rules were applied to determine boundaries of fields and the content of granaries to pyramids. Measuring of their lands was necessary for them because the Nile overflows and obliterates the boundaries between their properties every year. The lack of money or coinage

in the early centuries led to the development of a very detailed accounting system for the amounts of grain and beer stored in various places and paid out to the workers as salary.

1.7 Unit fractions

Dividing a loaf into fourths, there are four pieces, each a fourth part or $\frac{1}{4}$. Probably therefore the Egyptian computational practice allowed him only to admit the so-called unit fractions, that are fractions of the form 1/*n*, where *n* is a natural number. That was written by placing the sign ⬭ over the hieroglyphs for *n*. So, one indicated $\frac{1}{5}$ by $\overset{\frown}{5}$ and $\frac{1}{12}$ by $\overset{\frown}{12}$.
For some simple daily fractions there were descriptions as:
'the two parts' for $\frac{2}{3}$, 'the third part' for $\frac{1}{3}$, and
'the three parts' for $\frac{3}{4}$, 'the fourth part' for $\frac{1}{4}$.
With 'the third part' was meant the part you need to complete $\frac{2}{3}$ to a whole. For some of these fractions there were special symbols:
 or for $\frac{1}{2}$, X for $\frac{1}{4}$ and for $\frac{2}{3}$; $\frac{3}{4}$ had to be decomposed into $\frac{1}{2} + \frac{1}{4}$.

In *The Exact Sciences in Antiquity,* Neugebauer denotes $\frac{1}{n}$ by $\overline{n}$ and $\frac{2}{3}$ by $\overline{\overline{3}}$. From here this denotation will be used.
In the 'London Mathematical Leather Scroll' BM 10250 there are operations with unit fractions, as:

$\overline{6}+\overline{6}=\overline{3}$, $\overline{6}+\overline{6}+\overline{6}=\overline{2}$, $\overline{3}+\overline{3}=\overline{\overline{3}}$ and $\overline{3}+\overline{6}=\overline{2}$
If $\overline{6}$ are added to both of the terms of the last equation you will get:

$\overline{6}+\overline{6}+\overline{3}=\overline{2}+\overline{6}$ or $\overline{3}+\overline{3}=\overline{2}+\overline{6}$ or $\overline{\overline{3}}=\overline{2}+\overline{6}$
Adding again $\overline{6}$ to both of the terms you will get:

$\overline{6}+\overline{\overline{3}}=\overline{2}+\overline{6}+\overline{6}$ or $\overline{\overline{3}}+\overline{6}=\overline{2}+\overline{3}$, two equivalent denotations for $\frac{5}{6}$.
Adding $\overline{6}$ to both of the terms of $\overline{2}+\overline{3}+\overline{6}=1$, you will get $\overline{\overline{3}}+\overline{2}=1+\overline{6}$

Every Egyptian calculator had to know by heart these rules for calculating with halves, thirds and sixths. In the Rhind papyrus they are always brought in practice simplicity without further explanation.

1.8 Doubling of unit fractions

We already saw at $\overline{3}+\overline{3}=\overline{2}+\overline{6}$ a rule that can be brought in practice at each doubling of a unit fraction, of which the denominator is divisible by 3, e.g.:

$\overline{9}+\overline{9}=\overline{6}+\overline{18}$ or $\overline{15}+\overline{15}=\overline{10}+\overline{30}$ etc.

Why not simply maintain $\overline{9}+\overline{9}$? Suppose that this has to be redoubled again, than in that way it follows: $\overline{9}+\overline{9}+\overline{9}+\overline{9}$ etc. Therefore the Egyptian calculated as follows:
Denominators of fractions $\frac{2}{n}$ whose denominator is divisible by 2 were simply halved.
Fractions $\frac{2}{n}$ whose denominators are divisible by 3, all follow the general rule (knowing by heart): $\frac{2}{3k}=\frac{1}{2k}+\frac{1}{6k}$, e.g.: $\frac{2}{15}$ (k = 15 ÷ 3 = 5) : $\frac{2}{15}=\frac{1}{10}+\frac{1}{30}$, thus: 1 $\overline{9}$.
2 $\overline{6}+\overline{18}$
4 $\overline{3}+\overline{9}$.
8 $\overline{\overline{3}}+\overline{6}+\overline{18}$ etc.

Division was not always as simple as in the example given in § 1.6, and fractions would often have to be introduced.
To divide 19 by 8, problem Rhind no.24, the scribe would start by doubling the divisor, 8, to the point at which the next duplication would exceed the dividend, 19. Then he would start halving the divisor in order to complete the remainder. The calculation might appear thus:

	1	8
/	$\overline{2}$	16
	$\overline{2}$	4
/	$\overline{4}$	2
/	$\overline{8}$	1
Totals	$2 + \overline{4} + \overline{8}$	19

and Rhind nr 25: $16 \div 3$

/	1	3
	2	6
/	$\overline{4}$	12
	$\overline{\overline{3}}$	2
/	$\overline{3}$	1
Totals	$5 + \overline{3}$	16

also Rhind 21: $4 \div 15$

	$\overline{1}$	15
	$\overline{10}$	$1 + \overline{2}$
/	$\overline{5}$	3
/	$\overline{15}$	1
Totals	$\overline{5} + \overline{15}$	4

In Rhind no. 24 we read: $\overline{2}\ \overline{4}\ \overline{8}$ etc., in no. 25 this is $\overline{\overline{3}}\ \overline{3}\ \overline{6}$ etc. It is a peculiarity of the Egyptian calculators to start with $\overline{\overline{3}}$, not at once with $\overline{3}$. The sequence $\overline{\overline{3}}\ \overline{3}\ \overline{6}$ etc. obvious was fixed in tradition.

1.9 The (2/n) table [3)]

To multiply (or divide) numbers with fractions the Egyptians always had to double. For fractions whose denominator is not divisible by 2 or 3 they used the so-called (2/*n*) table (given in the Rhind Papyrus) for all odd values of *n* from 3 to 101. If we ignore the representations for fractions of the form $\frac{2}{3k}$, (which were given in the table) then the remainder of the (2/n) table looks like:

$2 : 3 = \overline{2} + \overline{6}$
$2 : 5 = \overline{3} + \overline{15}$
$2 : 7 = \overline{4} + \overline{28}$
$2 : 11 = \overline{6} + \overline{66}$
$2 : 13 = \overline{8} + \overline{52} + \overline{104}$
$2 : 17 = \overline{12} + \overline{51} + \overline{68}$
$2 : 19 = \overline{12} + \overline{76} + \overline{114}$
$2 : 23 = \overline{12} + \overline{276}$
$2 : 25 = \overline{15} + \overline{75}$
$2 : 29 = \overline{24} + \overline{58} + \overline{174} + \overline{232}$
$2 : 31 = \overline{20} + \overline{124} + \overline{155}$
$2 : 35 = \overline{30} + \overline{42}$

$2 : 37 = \overline{24} + \overline{111} + \overline{296}$
$2 : 41 = \overline{24} + \overline{246} + \overline{328}$
$2 : 43 = \overline{42} + \overline{86} + \overline{129} + \overline{301}$
$2 : 47 = \overline{30} + \overline{141} + \overline{470}$
$2 : 49 = \overline{28} + \overline{196}$
$2 : 51 = \overline{34} + \overline{102}$
$2 : 53 = \overline{30} + \overline{318} + \overline{795}$
$2 : 55 = \overline{30} + \overline{330}$
$2 : 59 = \overline{36} + \overline{236} + \overline{531}$
$2 : 61 = \overline{40} + \overline{244} + \overline{488} + \overline{610}$
$2 : 65 = \overline{39} + \overline{195}$
$2 : 67 = \overline{40} + \overline{335} + \overline{536}$

$2 : 71 = \overline{40} + \overline{568} + \overline{710}$
$2 : 73 = \overline{60} + \overline{219} + \overline{292} + \overline{365}$
$2 : 77 = \overline{44} + \overline{308}$
$2 : 79 = \overline{60} + \overline{237} + \overline{316} + \overline{790}$
$2 : 83 = \overline{60} + \overline{332} + \overline{415} + \overline{498}$
$2 : 85 = \overline{51} + \overline{255}$
$2 : 89 = \overline{60} + \overline{356} + \overline{534} + \overline{890}$
$2 : 91 = \overline{70} + \overline{130}$
$2 : 95 = \overline{60} + \overline{380} + \overline{570}$
$2 : 97 = \overline{56} + \overline{679} + \overline{776}$
$2 : 101 = \overline{101} + \overline{202} + \overline{303} + \overline{606}$

1.10 The red sub-numbers [3)]

How did the Egyptians calculate the values of the other fractions (not divisible by 2 or 3)? To calculate 2 x *n* they divided 2 by *n*, by simply carrying out the divisions, as shown in the next examples from the Rhind Papyrus, e.g.:

Which part is 2 of 31? ($2 \div 31$) $\overline{20}$ is $1 + \overline{2} + \overline{20}$ $\overline{124}$ is $\overline{4}$ $\overline{155}$ is $\overline{5}$

Calculation	1	31	($\overline{10}$ of 31 = $3 + \overline{10}$)
	$\overline{20}$	$1 + \overline{2} + \overline{20}$	How did they complete the fraction to 2?
	$\overline{124}$	$\overline{4}$	(or $\overline{2} + \overline{20}$ to 1) to find $\overline{4}$ and $\overline{5}$?
	$\overline{155}$	$\overline{5}$	

The numbers 20 124 and 155 immediately leap at the eye because they are written in red.
First one thought that the scribe had 'corrected' these numbers with red ink.
In every complicated division a fraction has to complete to 1 or 2.

By which is $\overline{\overline{3}} + \overline{15}$ completed to 1?

10 1 total 11, remainder 4 (15 – 11)

Calculate with 4 till you find 15? Or divide 4 by 15, see Rhind 21
10 and 1 are the 'numerators' of the fractions 3 and 15; in combination with denominator 15; their total is 11, ($^{11}/_{15}$) to obtain $^{15}/_{15}$ there fails $^{4}/_{15}$

Which part is 2 of 35? (2 ÷ 35) $\overline{30}$ is $1 + \overline{6}$ $\overline{42}$ is $\overline{\overline{3}} + \overline{6}$

6 7 5

Calculation $\overline{1}$ 35 How did they find $(1 + \overline{6}) + (\overline{\overline{3}} + \overline{6}) = 2$? The red 6 and
$\overline{30}$ $1 + \overline{6}$ the numbers 7 and 5 are reciprocally with 35, 30 and 42
$\overline{42}$ $\overline{\overline{3}} + \overline{6}$ thus directly proportional to the fractions $\overline{35}$, $\overline{30}$ and $\overline{42}$

In modern notation: $\frac{2}{35} = \frac{12}{210} = \frac{7}{210} + \frac{5}{210} = \frac{1}{30} + \frac{1}{42}$, this is equal to $\frac{2}{35} = \overline{30} + \overline{42}$

Which part is 2 of 5? (2 ÷ 5) $\overline{3}$ is $1 + \overline{\overline{3}}$ $\overline{15}$ is $\overline{3}$

that means $^{1}/_{3}$ part of $5 = 1 + \overline{\overline{3}}$ $^{1}/_{15}$ part of $5 = \overline{3}$ total 2

Calculation: $\overline{\overline{3}}$ 5 Two times $\overline{5}$ may represented as $\overline{3}$ of $\overline{5}$ or $\overline{15}$ plus a remainder,
$3 + \overline{3}$ which must complete the factor 2 and which is $1 + \overline{\overline{3}}$
$/\ \overline{3}$ $1 + \overline{\overline{3}}$ The question of finding $1 + \overline{\overline{3}}$ of 5 now arises. This is done
$/\overline{15}$ $\overline{3}$ by counting the thirds and writing their number in red ink.

Which part is 2 of 7? (7 ÷ 2) $\overline{4}$ is $1 + \overline{2} + \overline{4}$ $\overline{28}$ is $\overline{4}$

Calculation: $\overline{1}$ 7
$\overline{2}$ $3 + \overline{2}$ 1 7 Here is indicated
$/4$ $1 + \overline{2} + \overline{4}$ 2 14 how the 'denominator' 28
$/4$ 28 $\overline{4}$ $/4$ 28 was found

1.11 The area of a circle

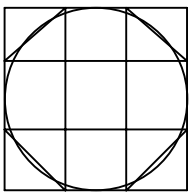


Figure 1.2

Problem 48 of the Rhind Papyrus is to find the area of an octogram obtained from a square of side 9 by dividing each side into thirds and connecting the points. The method is to find the area of the square $9^2 = 81$ and to subtract from that area the area of the four triangles in the corners 4 times (3^2x ½)=18. Thus the area of the octagon is 81 – 18 = 63. This octogram can be viewed as an approximation to a circle of diameter 9. Its area was found to be 63 which is almost that of a square of side 8.

In problem 50 of the Rhind Papyrus the area of a circle of diameter 9 is calculated as that of a square of side 8, a method, which may have been based on problem 48:
Example of a round field of diameter 9. What is the area? Take away ⅑ of the diameter; the remainder is 8. Multiply 8 times 8; it makes 64. Therefore, the area is 64.
If the scribe had inscribed a circle in the same square as done in Figure 1.2, he would have seen that its area was approximately that of the octagon.
How 'exact' this calculation is appears after checking with πr^2 : 3,14 x $(4,5)^2$ = 63,585 or the value of π = 3,16.
In two other problems, where a circle is part of a more complex situation, the same procedure was used.

1.12 The volume of a truncated pyramid in the Moscow Papyrus [3)]

The Moscow mathematical Papyrus, (2700 – 1670BC) was discovered early in the 1890's in the Necropolis of Dra Abu'l Negga. A Russian professor, W.Golenishchev, purchased it. After his death in 1912 his entire collection was acquired by the Moscow Museum of Fine Arts. The Papyrus consists of 25 mathematical problems and their solutions.
The Egyptians were not consistent in their use of a single symbol to denote a single concept. In the Moscow Papyrus a pair of legs walking 'into', (the adding sign), stands for the square of the associated number. The solutions are basically arithmetical, no trace of a proof, see problem no. 14: the calculation of the volume of a truncated pyramid.

If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top: You are to square this 4; result 16. You are to double 4; result 8. You are to square this 2; result 4.You are to add the 16 and the 8 and the 4; result 28; You are to take 1/3 of 6; result 2. You are to take 28 twice; result 56. See, it is 56. You will find it right.
See Figure 1.2 a and b

Figure 1.2 a A hieratic transcription of problem no. 14

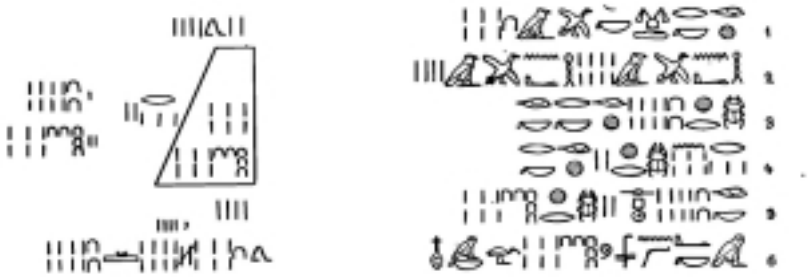


Figure 1.2 b A hieroglyphic transliteration of problem no. 14

The volume V of a truncated pyramid where h is the altitude and a and b are the lengths of the sides of the square base and the square top respectively is:

$$V = \frac{h}{3}(a^2 + ab + b^2)$$

The calculation is correct.

1.13 Linear equations with one unknown [4)]

A general technique for solving linear equations which is based on a method known as the 'false assumption'. Example (Rhind 24): A quantity and its 7th added become 19. What is the quantity? The scribe reasoned as follows: if the answer were 7 then $7 + \overline{7}$ of 7 would be 8. The number that 8 must be multiplied by to get 19 is $(2 + \overline{4} + \overline{8})$, (see §1.8, Rhind 24, 19 : 8) or: add till you get 19. Thus the quantity is $16 + \overline{2} + \overline{8}$.

	/ 1	$2 + \overline{4} + \overline{8}$
	/ 2	$4 + \overline{2} + \overline{4}$
	/ 4	$9 + \overline{2}$
totals	7	$16 + \overline{2} + \overline{8}$
	1	$2 + \overline{4} + \overline{8}$
	8	19

Example 2: *It is said that the area of a square of 100 (square cubits) is equal to that of two smaller squares. The side of one is $\overline{2} + \overline{4}$ of the other ($= \frac{3}{4}$). Let me know the sides of the two unknown squares.*

Solution 1: The symbolic algebraic approach. Let x and y be the sides of the two smaller squares. Than is $x^2 + y^2 = 100$ and $4x - 3y = 0$ or $4x = 3y$. From this you can solve x and y: Divide the last equation by 4: $x = \frac{3}{4}y$. Substitute x in de first equation by $\frac{3}{4}y$: $(\frac{3}{4}y)^2 + y^2 = 100$

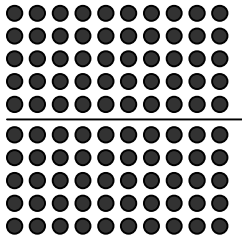
$\frac{9}{16}y^2 + \frac{16}{16}y^2 = 100 \qquad 25y^2 = 1600 \qquad y^2 = 64 \qquad y = 8$ and $x = 6$.

Solution 2: The Egyptian rhetoric algebraic approach:

Take a square of side 1 cubit (i.e. a false value of y equal to 1 cubit). Then the other square will have side $\overline{2} + \overline{4}$ cubit (i.e. $x = \overline{2} + \overline{4}$). The areas of the squares are 1 and $\overline{2} + \overline{16}$ square cubits respectively. Adding the areas of the two squares will give $\overline{1} + \overline{2} + \overline{16}$ square cubits. Take the square root of this sum: $\overline{1} + \overline{2} + \overline{16}$ gives $1 + \overline{4}$. Take the square root of 100 square cubits:10. Divide this 10 by this $1 + \overline{4}$. This gives 8 cubits, the side of one square. The side of the smaller square is $\overline{2} + \overline{4}$ times 8 cubits = 6 cubits.

Would the Egyptian scribe have been aware of the fact that the three squares of sides 6, 8 and 10 fulfil the Pythagoras' theorem?

Nowhere is found any trace of a deduction or a proof that makes clear that Egyptian arithmetic not only was for daily practice.



Although statues of scribes are found, there are no images known of calculating on some kind of a counting board. However on the reverse side of two papyrus documents two diagrams are found in the shape of a square of 10 times 10 dots, divided by a line between the fifth and sixth row. Perhaps an 'aide mémoire'. In both cases there was no connection with the text on the front side, see picture.

1.14 The Egyptian calendar

To predict the time of the start of the annually inundation by the Nile was very important for agriculture. The priests calculated the average period between the start of two overflows at 365 days. They found out that the start of the yearly inundation coincided with the appearance of the Dog Star Sirius in the sky just before sunrise. They saw this connection so often that they started to believe that this star causes the yearly overflow of the Nile. The firmament enabled the priests to divide the solar year into three seasons of four months of 30 days. The five remaining days became festive days.

1) H. L. Resnikoff / R.O. Wells Jr.
Mathematics in Civilisation Stone of Rosetta

2) Henrietta O. Midonick: The Treasury of Mathematics

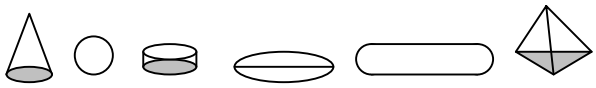
3) B. L. van der Waerden:
Science Awakening; Egyptian, Babylonian and Greek mathemathics
Much information about the use of the (2:*n*) table and the use of the red sub-numbers

4) George Gheverghese Joseph:
The Crest of the Peacock Non-European Roots of Mathematics

5) J. M. Pullan: The History of he Abacus

Chapter two, *Mesopotamia*

2.1 Plain Tokens [1)]



Mesopotamia, meaning the land between the rivers, (Euphrates and Tigris), the name given by the ancient Greeks. There, Neolithic farmers used about 8000 B.C. small hand-made clay counters: **Tokens**. Unlike tallies (see chapter 15), which were still in use well into the sixth millennium B.C., and recorded only quantitative information, the tokens conveyed also qualitative information. For instance, the type of item counted was indicated by the token shape while the number of units involved was shown by the corresponding number of tokens. These plain tokens, mainly of geometric shapes such as cones and spheres 1-3 cm. and also 3-5 cm., flat and lenticular disks, cylinders and tetrahedrons served for accounting animals, grain and land. The similarity of shape between counters indicating grain and land measures may point to a common usage in early societies to calculate land measurements in term of seed ratio necessary for sowing. Tokens were the first clay objects to have been fired into ceramic. Tokens were symbols, independent of phonetics, and people speaking different languages could share them. Plain tokens were widely used in the entire Near East and existed for five thousand years.

2.2 Complex tokens

Complex tokens (4400 – 3100 B.C.) were the consequences of urban development, the rise of industry, and the formation of the state. Complex tokens had a larger repertory of geometric shapes and also included more naturalistic forms. They bore a great variety of linear and punched markings. Some specimens were perforated in order to be strung for safekeeping in archives. After 3500 B.C. the major of complex tokens stood for finished products such as bread, oil, perfume, wool and rope, and for items produced in workshops such as pieces of furniture, tools and a variety of pottery vessels. During its entire duration the tokens were used by the elite to pool and redistribute communal resources. The role of tokens was accounting rather than mere computing. Tokens fulfilled a temple function rather than a domestic or commercial use.

2.3 Bullae

In the early fourth millennium B.C. two methods were devised to store tokens in archives. The first consisted of tying perforated tokens with a string. Both ends of the string were tied together and held by a solid ball of clay, about 7 cm in length and 5 in diameter, a **Bulla**. Then the bullae were covered with sealing. These sealing were made by a cylinder - seal.

2.4 Envelopes

The second method of storing tokens in archives consisted of enclosing the counters in clay **envelopes**. These envelopes consist of spherical, ovoid or oblong hollow clayballs. They measured about 5 – 7 cm. They were covered with seal impressions. A drawback of the envelopes was that they hid the enclosed tokens. From 3500 B.C. accountants resolved the problem by imprinting the shapes of the tokens on the surface of the envelopes prior to enclosing them. The number of units of goods was still expressed by a corresponding number of markings.

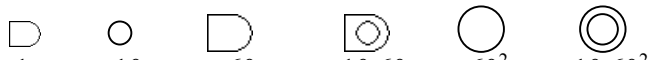
2.5 Clay tablets

The fourth millennium accountants ‘soon’ realised that the tokens within the envelopes were unnecessary in the presence of markings on the outer surface. Solid clay tablets bearing impressed signs replaced the hollow envelopes holding tokens. These impressed signs still perpetuated the shape of the tokens, but they assumed an entirely new function. Whereas the markings on envelopes repeated only the message encoded in the tokens held inside, the signs impressed on tablets *were* the message: they are the immediate forerunners of the Sumerian pictographic script.

2.6 Pictographs and Numerals

Figure 2.1

The accountants of Uruk about 3100 B.C. can be credited with creating numerals, they devised two types of signs: *numerals* (symbols encoding abstract numbers) and *pictographs* (expressing commodities). Pictographs were *incised*, whereas numerals were *impressed* in the still soft clay with a stylus, with a pointed and a blunt end, upright or oblique.
The pictograph for a sheep, ewe, or lamb was ⊕ ,⊕ or ◎ respectively. See Figure 2.1
The first numerals were not symbols specifically created for representing abstract numbers. Instead, they were the impressed signs, formerly indicating animals or units of goods. The wedge, which originally meant a small quantity of grain, now stood for 1; the circle, which represented a larger quantity of grain, was 10; the large wedge, punched wedge, and large circle were greater numbers:



1	10	60	10.60	60^2	10.60^2
aš	u	geš	geš-u	šar	šar-u

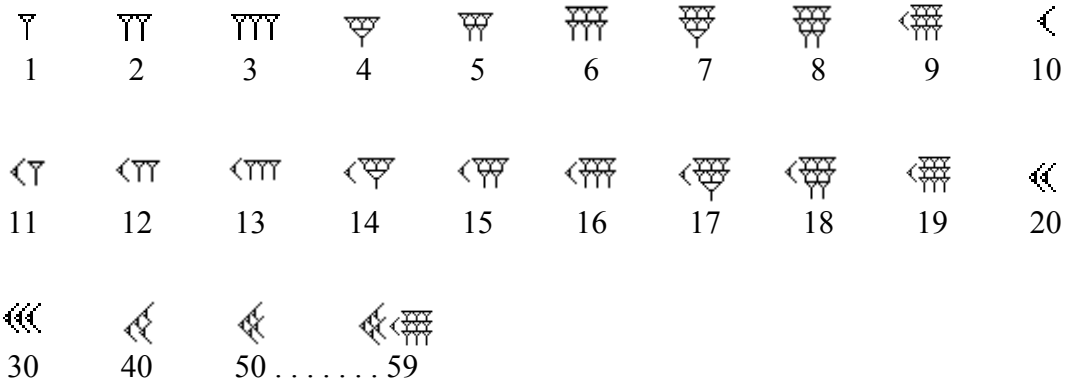
There is yet another possible alternative, as follows. The impressed units of animal numeration could also have served as numerals: the long wedge standing for 1 and the circular marking, corresponding to the former lenticular disk, for 10. In this case, the animal numeration would have produced two units of the Sumerian arithmetical system: 1 and 10. Animals would have been counted by ten, probably using the fingers. This in turn could explain the mixed decimal-sexagesimal Sumerian counting system. It could be that 6 derived from the former unit of grain metrology (sphere). On the other hand, 60 was particularly convenient for grain metrology, having the unique property of being divisible by
1, 2, 3, 4, 5, 6, 10, 12, 15, 20 and 30.
The presentation of signs on a tablet, starting with the largest units at the top or on the right and proceeding in hierarchical order, probably perpetuated the way tokens were organised.
The invention of abstract numerals was the beginning of mathematics, and was also the beginning of writing.

Tallies, tokens, and writing reflected three major phases in the development of counting:
1) one to one correspondence,
2) concrete counting,
3) abstract counting.
Writing is the outcome of abstract counting.
Later writing would be done with a more functional triangular stylus. Now it was possible to record and communicate the name of the sponsors/recipient of the merchandise formerly indicated by seals.

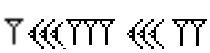
The Babylonian used a stylus with a triangular profile. With this they made impressions in moist clay. The sharp edge made a vertical stroke | and the base a more deep impression ▼, so that the combined effect was a head and tail figure resembling a wedge ▼.
Because the Latin word for wedge is *cuneus*, the resulting style of writing has become known as *cuneiform*.
Two basic cuneiform symbols and formed the characters in the Babylonian writing system as well as their numbers.

2.7 Cuneiform number system

First the description of their notation. The two numerical symbols and
correspond to the *one* and *ten* of a primitive decimal system. Numbers smaller than 60 were formed in a straight forward fashion [2]:



But what follows is based on a sexagesimal system:
60 61(60+1) 70(60+10) 80(60+20)

An example in this system:  written in Arabic numerals:

$$1.60^2 + 33.60^1 + 32.60^0 = 3600 + 1980 + 32 = 5612$$

Following Neugebauer [3], we represent a Babylonian sexagesimal number by the following scheme: …a,b,c;d,e,… where a,b,c, are decimal numbers from 1 to 59 and the semicolon represents the sexagesimal point, which separates the integral part of the non-integral part just as the decimal point does in the decimal system.
For instance: 3,1;5 represents the number 3x60 + 1 + 5/60 = 181 1/12
The remarkable achievement of the Babylonians, one of the most distinguished in antiquity, was that with these two symbols now they were able to systematically represent arbitrarily large numbers and arbitrarily small fractions.

For some numbers there were variants:
for 4: for 7: and for 8:

19 Could be written as: that should be read as: 20 - 1

There also were special signs for fractions if they represent a number (for quantities):
½= 1/3= 2/3= and 5/6 =

Their system however lacked two basic features of our modern decimal system.
First there was no ‘sexagesimal point’;
the symbol could also mean 3600, 1/60 or 1/3600 etc. depending on the context:
is this $2.60 + 24 = 144$ or $2.60^2 + 24.60 = 8640$ or $2 + \frac{24}{60} = 2{,}4$?

Second, there was no zero. A gap was often used to indicate that a whole sexagesimal place was missing, but this rule was not strictly applied and confusion could result:
is this $60 + 24$ or maybe $60^2 + 0 + 24$?

Later on, from 300 B.C. the special symbol was introduced to denote an unfilled position as in: $= 2.60^2 + 0.60 + 24 = 7224$
The confusion was not ended, since this Babylonian divider was used only medially and there still existed no symbol to indicate the absence of a digit at the end of a number.
Their numerical representation expressed the relative order of the digits, and the context alone decided the magnitude of a sexagesimally written number; since the base was so large, it was usually evident what value was intended.

The very same tablet which contains hundreds of sexagesimal numbers, to compute the dates of the new moons for a given year, might end with a ‘colophon’ containing the name of the owner of the tablet, the name of the scribe, and the date of writing of the text, the year being expressed in the form 2 *me* 25 ‘2 hundred 25’, where the main text would express the very same date sexagesimally as 3,45. In other words, it is only in strictly mathematical or astronomical contexts that the sexagesimal system is consistently applied.

2.8 Multiplication, tables and proceedings

Most of the mathematical tablets have been classified as ‘table texts’ or as ‘problem texts’; they were ever-present aids both for instructional problems and for practical use. They included tables for reciprocals, squares and square roots, cubes and cube roots as well as multiplication tables. For instance two tablets from ca. 1950 B.C. (First Babylonian Dynasty) with multiplication from 1x1 to 59x59 !, the squares from 1 to 59 and the cubes from 1 to 32.
Arithmetic was not ‘algebraic’ that means based on manipulating of symbols in formulas but ‘algorithmic’ that means following special proceedings to get the solution of a problem.
An example of the solution of a problem from ca. 1750 BC:

Length, width. I have multiplied length and width, thus obtaining the area. Then I added to the area, the excess of the length over the width: 3,3 (i.e., 183 was the result) Moreover, I have added length and width: 27. Required length, width and area.

One follows this method: *27 + 3,3 = 3,30*	3 x 60 + 30 = 210
2 + 27 = 29	= 29
Take one-half of 29 (this gives 14;30)	14,5
14;30 x *14;30 = 3,30;15*	14,5 x 14,5 = 210,25
3,30;15 - 3,30 = 0;15	210,25 - 210 = 0,25
The square root of 0;15 is 0;30	$\sqrt{0{,}25}$ = 0,5
14;30 + 0;30 = 15 = length	14,5 + 0,5 = 15
14;30 - 0;30 = 14 = width	14,5 - 0,5 = 14

Subtract 2, which have been added to 27, from 14, the width 12 is the actual width
I have multiplied 15 length by 12 width
15 x *12 = 3,0 area* *15 - 12 = 3* *3,0 + 3 = 3,3*

Second problem:

Length + width = 50 Area = 600 What are length and width?

Step 1: take (l + w) ÷ 2 = 25

2: take the square of (l + b) ÷ 2 = 625

3: subtract from these 625 the area: 625 - 600 = 25

4: the square root of 25= 5

length = (l + w) ÷ 2 + 5 = 30 width = (l + w) ÷ 2 – 5 = 20

2.9 Divisions: the Table of Reciprocals [4)]

When in a problem a division a:b had to be carried out, the Babylonians used the rule that to divide by a number is equal to multiply by the reciprocal of that number. They were able to write 1/25 = ;2,24 and therefore, they could write 23 x 1/25 = (23)x(;2,24) = ;55,12 where we have carried out the multiplication in sexagesimal form. Consequently, most fractions posed no problem at all for them. Here is a

Table of Reciprocals of Standard Type (CBS 29 13 21)

1 ÷ 2	=	30	16	3,45	45	1,20
3		20	18	3,20	48	1,15
4		15	20	3	50	1,12
5		12	24	2,30	54	1,6,40
6		10	25	2,24	1	1
8		7,30	27	2,13,20	1,4	56,15
9		6,40	30	2	1,12	50
10		6	32	1,52,30	1,15	48
12		5	36	1,40	1.20	45
15		4	40	1,30	1,21	44,2640

The above 'table of reciprocals' is a list of numbers, b and 1/b, were 1/b are the reciprocals of the numbers b, such that the products b x 1/b = 1 *or any other power of 60*.

In this table 1 ÷ 2 = 30 means ½ = 0;30 ⅓ = 0;20 etc. It is irrelevant whether we write 2 x 30 = 1,0 or 2 x 0;30 = 1 or 0;2 x 30 = 1 or 0;2 x 0;30 = 0;1 etc. The Babylonians made full use of the flexibility of this system.

Tables of reciprocals in combination with tables of multiplication formed a complete system. Note that the table omits the reciprocals of certain numbers, such as 1/7 1/11 1/13 etc. The table laconically remarks "7 does not divide", "11 does not divide", etc. This holds true for all numbers which contain prime numbers different from 2, 3, and 5, smaller than 60. Therefore for instance 1/13 = 7/91 = 7(1/91) ≈ 7(1/90)

Problem: divide 1029 by 64. Or multiply 1029 by $\frac{1}{64}$.

In sexagesimal numbers: $1029 = 17.60^1 + 9.60^0$ or following Neugebauer 17,9.

Also $\frac{1}{64} = 56.60^{-2} + 15.60^{-3}$ or 0;0,56,15 (see table of reciprocals).

17,9 x 0;0,56,15 = 16;4,41,15 multiplied in the modern way:

```
   0;0,56,15
        17,  9
   ----------
     8,26,15
  15;56,15
  ----------
  16;  4,41,15
```

The result 16;4,41,15 can be read in Hindu-Arabic numerals as:

$16.60^0 + 4.60^{-1} + 41.60^{-2} + 15.60^{-3} = 16{,}0781$

Exactly as we multiply two numbers regardless of the position of the decimal point, one can also operate with the Babylonian numbers and determine the absolute value at the end if necessary. For the numerical process itself it is indeed a great advantage that one does not need to worry about special values for fractions and integers. It is precisely this feature which gave the Babylonian system its tremendous advantage over all other number systems in antiquity.

2.10 'Pythagorean triples' on Plimpton 322

Cuneiform tablets with mathematical contents are known to us mostly from the so-called 'Old Babylonian' period, between 1900 and 1600 B.C. They already show the highest level ever-attained in Babylon. The oldest preserved document in number theory is contained in a tablet known as **Plimpton 322** (Columbia University Libraries). See Figure 2.2.

Figure 2.2 Plimpton 322

In its present state, the tablet represents the right-hand part of a larger text. The size of the preserved part is 127 by 88 mm. This tablet, deciphered by Neugebauer, tabulates the answers to a problem containing Pythagorean numbers or triangles whose sides are integers. On the missing part stood most probably the value h: $h^2 + b^2 = d^2$. In the subjoined transliteration these values are added on the most right-hand side, d and b are given for 15 values.

Line	$\frac{b^2}{h^2}$	b (width)	d (diagonal) (hypotenuse)	No.	$h=\sqrt{d^2-b^2}$
3	[59,0,]15	1,59	2,49	1	2,0
4	[56,56,]58,14,50,6,15	56,7	1,20,25	2	57,36
5	[55,7,]]41,15,33,45	1,16,41	1,50,49	3	1,20,0
6	53,10,29,32,52,16	3,31,49	5,9,1	4	3,45,0
7	48,54,1,40	1,5	1,37	5	1,12
8	47,6,41,40	5,19	8,1	7	45,0
10	41,33,45,14,3,45	13,19	20,49	8	16,0
11	38,33,36,36	8,1	12,49	9	10,0
12	35,10,2,28,27,24,26,40	1,22,41	2,16,1	10	1,48,0
13	33,45	45	1,15	11	1,0
14	29,21,54,15	27,59	48,49	12	40,0
15	27,0,3,45	2,41	4,49	13	4,0
16	25,48,51,35,6,40	29,31	53,49	14	45,0
17	23,13,46,4[0]	28	53	15	45

Transliteration of Plimpton 322, including the missing part.

The heading b and d represent terms meaning ‘width’ and ‘diagonal/hypotenuse’. It is easy to verify the remarkable fact that the numbers in the third (b), fourth (d), and sixth (h) columns are all Pythagorean triples. On the tablet itself (Figure 2.2) only b and d are given explicitly, but h is given in terms of the ratio b^2/h^2. We have listed the corresponding values of $h=\sqrt{d^2 - b^2}$ (which are integers!) in the rightmost column. For instance:
$(2,0)^2 + (1,59)^2 = (2,49)^2$, or in base 10 representation, $(120)^2 + (119)^2 = (169)^2$ (first line).

2.11 Application of Plimpton 322

An old Babylonian text (on BM 85 196) reads as follows:

A patŭ (beam?) of length 0;30 (stands against a wall) = d 0,5
The upper end has slipped down a distance 0;6. 0,1
How far did the lower end move?
h = 0;30 - 0;6 = 0;24 0,5 - 0,1 = 0,4
than b is (according to ‘Pythagoras’) = $\sqrt{d^2 - h^2}$
and looked up in a table.

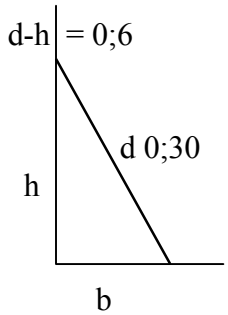


Figure 2.3

The values of Plimpton 322 in Arabic numerals:

Line	$(b:h)^2$	b = width	d = diagonal	No	$h = \sqrt{d^2-b^2}$
3	1,9834028	119	169	1	120
4	1,9491586	3367	4825	2	3456
5	1,9188021	4601	6649	3	4800
6	1,8862479	12709	18541	4	13500
7	1,8150077	65	97	5	72
8	1,7851929	319	481	6	360
9	1,7199837	2291	3541	7	2700
10	1,6845877	799	1249	8	960
11	1,6426694	481	769	9	600
12	1,586122	4961	8161	10	6480
13	1,562500	45	75	11	60
14	1,4894168	1679	2929	12	2400
15	1,4500174	161	289	13	240
16	1,4302388	1771	3229	14	2700
17	1,3871605	56	106	15	90

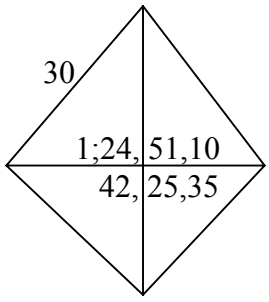
One interesting point concerning the triples on Plimpton 322 is that the right triangles corresponding to the sides (b,d,h) have nearly uniformly decreasing values of the ratio $(b:h)^2$ (the first column), which means that the angle a subtended by the sides d and h is decreasing at a nearly uniform rate as one proceeds the rows of the tablet. Perhaps the Babylonian mathematician who worked out this table purposefully desired to construct a table of values of the tangent function. The right triangle in line 3 is nearly 90° 45° 45°; the last is nearly 90° 60° 30°.

The Babylonians also could work with right triangles whose sides were not rationally related. For instance $\sqrt{2}$ (see YBC 7289, Yale Babylonian Collection), which contains the diagram and numbers given in Figure 2.4.

2.12 The root of 2 or $\sqrt{2}$

YBC 7289

Transcription
Figure 2.4



Transliteration in
Arabic numerals

The number 1,24,51,10 is, if we assume that the 'sexagesimal point' occurs between 1 and 24 (that is 1;24,51,10) = $1 + \frac{24}{60} + \frac{51}{60x60} + \frac{10}{60x60x60} = 1 + \frac{2}{5} + \frac{51}{3600} + \frac{10}{21.600} = 1 + 0.400000 + 0.0141666 + 0.0000462 = 1.4142129$ while $\sqrt{2} = 1.414213562$ that means about 0.0000007 too small! It is easy to check that this interpretation is correct, for $30\sqrt{2}$ (see top left) multiplied with 1;24,51,10 gives added up: 30 x 1 = 30 30 x ;24 = $\frac{30x24}{60}$ =12
30 x ;0,51 = 30 x $\frac{51}{60x60} = \frac{51}{2x60} = \frac{25}{60} + \frac{0,5}{60} = \frac{25}{60} + \frac{30}{60x60} =$;25,30 30 x $\frac{10}{60.60.60} = \frac{1}{2}\left(\frac{10}{60.60}\right) = \frac{5}{60.60} =$;0,5
added up 42;25,35 written under the diagonal.

2.13 Astronomy

The second period from which we have a large number of mathematical texts is the latest period of Babylonian history, when Mesopotamia had become a part of the empire of Alexander's successors, the *Seleucids*. This period, from about 300 B.C. has furnished us with a great number of astronomical texts of a most remarkable mathematical character. With the use of methods of approximating the values of certain irrational numbers, such as $\sqrt{2}$, later Babylonian mathematicians constructed tables of the lengths of chords of a circle of a given radius, which correspond to modern tables of sine's and were particularly useful for astronomical purposes.
Babylonian cosmology (in effect) placed the earth at the centre of a sphere to which the stars were fixed and the motions of the sun, moon, and planets were studied in relation to the background of stars which served then, as it does now, as a frame of reference. The basic periodic motions of the sun, moon, and the five planets, visible to the unaided eye, were measured. The Babylonian astronomers established an arithmetic model, capable of predicting their conjunctions, and (partial) eclipses, and other important features.
Astronomical observations also formed the basis of a calendar.

1) Denise Schmandt-Besserat
Before Writing, From Counting to Cuneiform

2) Via Internet at http://it.stlawu.edu/~dmelvill/mesomath/Numbers.html

3) O. Neugebauer: The Exact Sciences in Antiquity

4) H. L. Resnikoff: Mathematics in Civilisation

Chapter three, *The Greeks*

3.1 The Thalassic Age

The interval from roughly 800 B.C. to 800 A.D. is sometimes known as the Thalassic Age, that is the 'sea' age. The first part of the Thalassic Age is labelled the Hellenic era to indicate the source of the new inspiration: the Dorians, the unlettered invaders from the north. They brought with them no mathematical or literary tradition, but, at the shores of the Mediterranean, they took over the intellectual leadership from the valleys of the Nile, Tigris, and Euphrates rivers.
The first Olympic Games were held in 776 B.C., and by that time a wonderful Greek literature already had developed, evidenced by the works of Homer and Hesiod.
During the 6th century B.C. there appeared two men, Thales and Pythagoras. No mathematical masterpiece from either one has survived, or it is even established that either Thales or Pythagoras ever composed such a work. What they may have done must be reconstructed.

3.2 Thales of Miletus (ca. 624 to 548 B.C.)

Thales as well as Pythagoras were in close proximity to the two chief river valleys from which knowledge could be derived and they were in a position to travel to centres of ancient learning and there acquire first hand information on astronomy and mathematics. In Egypt they are said to have learned geometry; in Babylon, under the enlightened Chaldean ruler Nebuchadnezzar, Thales probably came in touch with astronomical tables and instruments. We know that a large body of mathematical material was familiar to the Babylonian a millennium before the time of Thales. Tradition has it that in 585 B.C. Thales amazed his countrymen by predicting the solar eclipse of May 28 of that year. In any case, Thales is the first man in history to whom specific mathematical discoveries have been attributed.

3.3 Pythagoras of Samos (ca. 580 – 500 B.C.) and the Pythagorean school

Pythagoras also travelled to Egypt and Babylon – possibly even to India. During his peregrinations he evidently absorbed not only mathematical and astronomical information, but also much religious lore. Pythagoras was, incidentally, virtually a contemporary of Buddha, of Confucius, and of Lao-Tze.
When he returned he settled at Croton (South-eastern coast of Magna Graecia – now Italy). There he established a secret society. This so-called Pythagorean school of thought was political conservative and with a strict code of conduct. Vegetarianism was enjoined upon the members, apparently because Pythagoreanism accepted the doctrine of metempsychosis, or transmigration of souls. Attributions of discoveries were not to be made to a specific member of the school, but rather to the contributions of the Pythagoreans. To the motto of the Pythagorean school is said to have been: "All is number". The Pythagoreans carried number worship to its extreme, basing their philosophy and their way of life upon it. Arithmetic now could be thought as an intellectual discipline as well as a technology. They tended to divide mathematics into two separate studies: *logistic* was the art of calculation, and *arithmetic* was what we would now call number theory. *Arithmetic* studies were considered to be important, while *logistic* was considered a 'vulgar and childish art'.

Examples of arithmetic ciphering are the three following series:

1	2	3	4	5	6	7	8	all figures / numbers
1	3	5	7	9	11	13	15	odd figures / numbers

The sum of the sequence of odd numbers beginning with 1 always gives a square number:

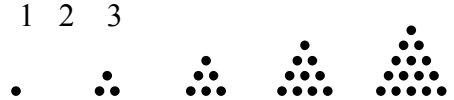
$1 + 3 = 4 = 2^2$ $\quad 1 + 3 + 5 = 9 = 3^2$ $\quad 1 + 3 + 5 + 7 = 16 = 4^2$

the successive sums of the odd figures always are cubes:

1=	3 + 5 =	7 + 9 + 11 =	13 + 15 + 17 + 19 =
1^3	$8 = 2^3$	$27 = 3^3$	$64 = 4^3$

They also considered numbers as generated by dots, and so classified numbers as linear (prime): • •• ••• etc. and as plane and solid, with respectively two and three factors:

1 2 3



1 3 6 10 15

a triangular number is one, which can be built up as the sum of the sequence 1 3 6 10…etc. geometrically, and

1 4 9

similarly, the square numbers are built up as sums of terms beginning with 1 in the series 1 4 9 etc. geometrically.

The Greeks employed common fractions as well as unit– and sexagesimal fractions, writing the numerator with one accent mark and the denominator written twice with two accent marks. Unit fractions were indicated simply by the denominator with one accent, quite similar to the Egyptian procedure. Occasionally the Greeks wrote the numerator with the denominator in the position where we write an exponent.

The very theorem to which the name of Pythagoras still clings quite likely was derived from the Babylonians. It is reasonable to assume that the earliest members of the Pythagorean School were familiar with geometric properties known to the Babylonians. Pythagoras proved 'his' theorem, which was in relation to the area of three squares, not in relation to the lengths of the three legs of a right-angled triangle. Pythagoras' theorem was known as well in China as in India and was used there in the *Soelva Soetras*, see chapter 6.

The Hellenes were celebrated as traders and businessmen, and there must have been a lower level of arithmetic or computation that satisfied the needs of the vast majority of Greek citizens. What evidence we have is usually in the form of quotations from ancient writers.

As early as the end of the 7th century B.C., **Solon**, the wise lawgiver of Athens, compared a tyrant's favourite to a counter whose worth depended entirely on the whim of the person who pushed it from one place to another.

The comic dramatist **Aristophanes** (ca. 445 to 385 B.C.) does say once in **The Wasps**: *First the income of the city of Athens is to be counted, and the expenditures of the judicial bench, "not indirectly by using counters, but right on the fingers".*

Demosthenes (ca. 384 to ca. 322 B.C.) wrote of the need to use pebbles for calculations that are too difficult to do mentally.

The historian **Polybius** (210 to 128 B.C.) made Solon say: *They who live at the courts of kings are as the pebbles on a counting-board. It is the will of the calculator who makes them be worth a chalkos or a talent.*

And last but not least, the Darius Vase:

3.4 The Darius Vase

This magnificent ceremonial vase in the red-figured Attic style, which commemorates the Persian Wars, was found in 1851 in an ancient grave near Canosa in Apulia (Southern Italy), and is now in the Museo Nationale in Naples, see Figure 3.1. This very large vase, about 130 cm tall, is covered with pictures. In the top row of the three rings of figures, Zeus and the other gods take Hellas, menaced by the Persians, under their protection. In the middle row, the Persian King Darius is seated on his throne. Arranged about Darius are figures symbolising the members of his court and officials of his government, including, in the bottom row, the revealing representation of Darius' royal treasurer, see figure 3.2.

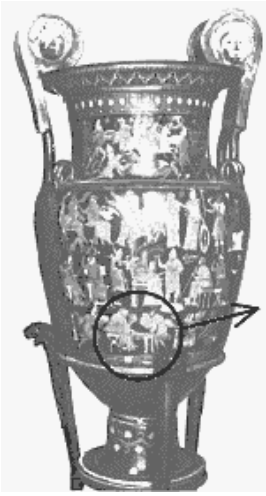


Figure 3.1

Figure 3.2

A man dressed in Persian costume stands in front of the table with a sack of coins. He is bringing in the tribute paid by the various conquered peoples, who are represented by the individual figures kneeling behind him. The treasurer is holding a wax tablet, possibly for recording the tribute, in one hand, while he seems to be manipulating some *psephoi*, pebbles or counters on a table with the other hand.

3.5 The Salamis counting board and the Attic or Herodianic numerals

The most complete, and best known, of the Greek counting boards is that found on the island of Salamis just a few miles off the Greek coast near Piraeus. This *abákion*, the Greek word for abacus table, of white marble, 149 cm long, 75 cm wide and varying in thickness from 4.5 to 7.5 cm at the edges, see Figure 3.3. Two groups of parallel lines have been chiselled into this abacus, one group of eleven lines crossed by a perpendicular line through their middle, and another group of five shorter ones. Along the two long sides of the table and across one of the shorter sides are symbols, which can be identified as early Greek numerals and also as denominations of coins, ranging from 1 to 1000 and in another from one-eight of an obol to 6000 drachmae. Drachmae and obols were units of weights as well as money in ancient Greece. The abacus is of a convenient size for use with pebbles, which could have been placed between the lines as

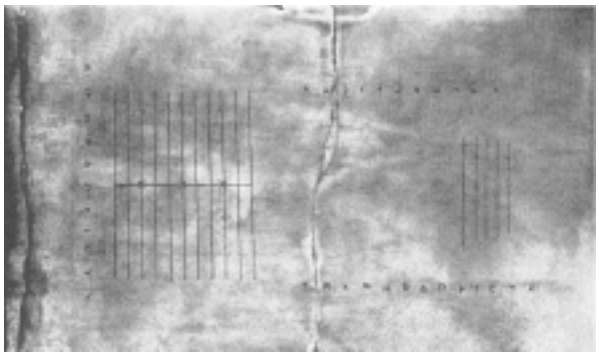
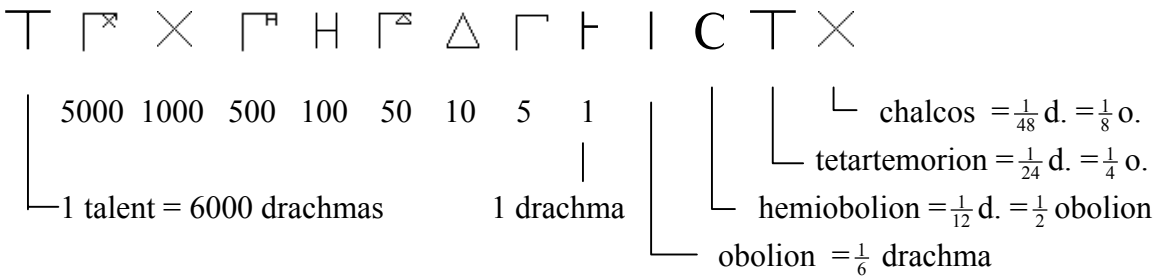


Figure 3.3

well as opposite the symbols. These numerical symbols are called Attic or Herodianic and were in use by the Greek from about 450 t0 95 B.C. The Attic system was named after Attica, the area around Athens, where it chiefly was used. In this more primitive system the numbers from one to four were represented by repeated vertical strokes. For the number five a new symbol – the initial letter Π or Γ of the word five, pente – was adopted.
For the numbers from six to nine, the Attic system combined the symbol Γ with unit strokes. For integral powers of the base (ten) the initial letters of the corresponding number words were adopted. The letter denoting 5 was combined with other letters to get intermediate symbols for 50, 500, 5,000 and 50,000:

	= 1		= 500 = chilioi
	= 5 = pente pi		= 10^3 = 1000 = chilioi
	= 10 = deka (delta)		= 5 x 10^3 = pentechilioi
	= 50 = pentedeka		= 10^4 = myrioi
	= 100 = hekaton		= 5 x 10^4 = pentemyrioi

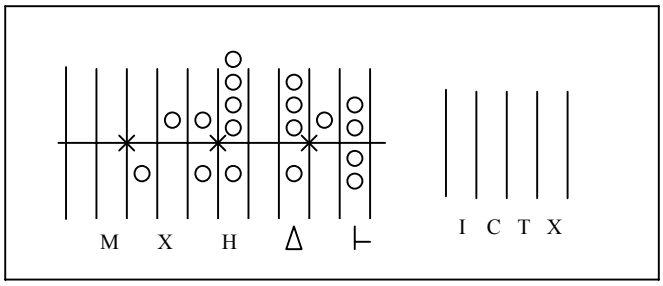
The numerical values of the Attic symbols on the Salamis table:



Obolion, hemiobolion, tetartemorion and chalcos are fractions of coins and weights. The The counters for the fractions were placed between the five short lines at the right-hand side. Here we see how ancient civilisations avoided an excessive use of fractions: they simply subdivided units of length, weight, and money so effectively that they could calculate in terms of integral multiples of the subdivisions. This undoubtedly is the explanation for the popularity in antiquity of duodecimal and sexagesimal subdivisions.

The columns indicated the place value of the counters. Thus the right-most column represented units, and each second column to the left increased by ten times the value of the second column to the right; also each space between those columns counted for five times that of the column to the right. No more than four counters could be placed in a 'ten' column and no more than one in a 'five' space. As soon as five counters appeared in a 'ten' column, they were removed and one counter placed in the next higher space; if two appeared in a 'five' space, they were removed and one placed in the next higher column. To perform an addition, e.g., 1937 + 5612, the first number is placed above the horizontal line, the second below that line,

10^3 900 30 7 = 1937 (upper counters)



5.10^3 600 10 2 = 5612 (lower counters)

Figure 3.4

then the counters in each column are grouped together according the above mentioned rules, see Figure 3.4. To perform a multiplication, the counters of one number are to be placed under the Attic symbols at one long side, the other under the symbols at the short side. For performing a multiplication see a comparable operation in chapter four.

3.6 The Ionic or Alexandrian numerals

The Greeks of Iona (the Greek part of Asia Minor, the old name of the Asiatic part of the present-day Turkey) also developed a ciphered numeral system, but with a more extensive set of symbols to be memorised. In this Ionic or Alexandrian system they ciphered their numbers by means of the 24 letters of the ordinary Greek alphabet, augmented by 3 obsolete Phoenician letters (the digamma ς for 6, the koppa ϟ for 90 and the sampi ϡ for 900). The first nine letters were associated with the numbers from 1 to 9; the next nine letters represented the first nine integral multiples of 10; the final nine letters were used for the first nine integral multiples of 100. First only capital letters were used, after the introduction of small letters in Greece, the association of letters and numbers appeared as follows:

α = 1	ζ = 7	μ= 40	ρ = 100	ψ = 700	An accent mark placed to the
β = 2	η = 8	ν= 50	σ = 200	ω = 800	left and below the appropriate
γ = 3	θ = 9	ξ = 60	τ = 300	ϡ = 900	unit multiplied the
δ = 4	ι = 10	ο = 70	υ = 400	,α = 1,000	corresponding number by
ε = 5	κ = 20	π = 80	φ = 500	,β = 2,000	1,000:
ς = 6	λ = 30	ϟ = 90	χ = 600	,γ = 3,000 etc.	

The symbols were always arranged in the same order, from the highest multiple of 10 on the left to the lowest on the right. To distinguish the numeral meaning of letters from their ordinary use in language, the Greeks added an accent at the end or a bar extended over them; thus the number 5612 might appear as ,ε χ ι β' or ,ε $\overline{\chi\iota}$ β = 5000 + 600 + 10 + 2.
Tens of thousands were indicated by using a new letter M, from the word myrioi, meaning ten thousand. The letter M placed next to the symbol(s) for a number 1 to 9,999 caused the numbers to be multiplied by 10,000, as with δ M = 40,000 or ψ ξ ε M ,δ τ κ α = 7,654,321

A drawback compared to the Attic-system is that there are more symbols to be memorised. An advantage is the short manner of writing, especially important on places where is too little room, as on coins, e.g.: ΩΜΘ instead of ΓΗΗΗΔΔΔΔΓΙΙΙΙ = 849

Multiplication in Greek alphabetic numerals was performed by beginning with the highest order in each factor and forming a sum of partial products, for example, 42 x 25:

χ ε	25	40 x 20 = 800
μ χ	42	2 x 20 = 40
ω μ	800 40	40 x 5 = 200
σ ι	200 10	2 x 5 = 10
,α ν	1000 50 = 1,050	

For performing a multiplication the Greeks used tables:

β	α	β
	β	δ
	γ	ς

χ	ι	σ
	χ	ν
	λ	χ

σ	ρ	β
	σ	δ
	τ	ζ

etc. or:

2 x 1 = 2	20 x 10 = 200	200 x 100 = 20,000
2 = 4	20 = 400	200 = 40,000
3 = 6	30 = 600	300 = 60,000 and so on to 10,000

This table was still published in a Greek arithmetic book of the 14th century!
They also used the Egyptian systems of doubling and halving.

3.7 Astronomy [2)]

In remote times the stars and other celestial bodies were worshipped as gods and the study of their motion formed part of the religious duties owed them. Then astronomy was the sole means of regulating the calendar and thereby determining the proper time for annual agriculture activities. The motion of the bodies in the solar system is viewed against the background of the fixed stars. The various star clusters were identified with familiar creatures and objects whose forms appeared to be similar to the patterns traced by the stars. Of principal importance were those constellations that lie, when viewed from the earth, behind the paths of motion of the planets, moon and sun. Since all these bodies move in nearly the same plane – the plane of the *eclipse* – their motion against the background of stars appears to take place in a relatively narrow band, the *zodiac*. The zodiac is partitioned into 12 equal *signs*, each comprising 30° in the *ecliptic* plane; each sign is associated with a constellation that lies in the zodiacal band from which the sign draws its name, in the order: Aquarius, Pisces, Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius and Capricorn. This system first appears in Babylon around the middle of the fifth century B.C.

Meton (* 460 -...), mathematician and astronomer, established that 19 solar years are nearly equal to 235 synodic-months or moon-months.

Once again Pythagoras. The Pythagoreans seem to have made arithmetic as the basis of a unification of all aspects of the word about them. They associated number with geometric extension; this in turn led them to an arithmetic of the heavens. The postulate of uniform circular motion that the Pythagoreans adopted was to dominate astronomical thought for more than 2000 years. The Pythagoreans explained celestial phenomena by way of number. Neither the Babylonians nor the Pythagoreans, however, developed a model to connect the various celestial phenomena.

Plato (427 to 347 B.C.), first astronomer. The basic model of the heavens developed at the time of Plato's Academy is one of two concentric spheres, the sphere of the earth and the sphere of the stars. Observations indicating that the shadow of the earth on the moon during a lunar eclipse has a circular edge convinced the Greeks of the earth's sphericity. Their sense of aesthetics – that a sphere was the most perfect solid shape – added to this conviction. That the shape of the heavens should mirror the shape of the earth was also only natural.
With the earth considered immovable, the observed daily motion in the sky must be due to the rotation of the celestial sphere to which were firmly attached the so-called fixed stars, grouped into patterns called constellations. These stars form the fixed background for the seven 'wanderers', the sun, the moon and five planets. These wanderers were more loosely attached to the celestial sphere. Sometimes they had their own motion, usually in the opposite direction. The astronomers were limited in their attempts at solution. Since the universe outside the earth was thought to be unchanging and perfect, according to Aristotle, the only movements in the heavens were the 'natural' movements of these perfect bodies. Because the bodies were *spherical*, the natural movements were *circular*. Because the basic Greek model of the heavens consisted of spheres, the first element of the study of celestial motion was the study of the properties of the sphere by astronomer- mathematicians.

To solve Plato's problem was to develop a model, which would explain the phenomena in the heavens through a combination of geometrical constructions using circular and uniform motion.

Eudoxus (400 to 345 B.C.), was largely responsible for turning astronomy into a mathematical science. He was the inventor of the modifications on the two-sphere model necessary to account for the various motions of the sun, moon, and planets, nevertheless keeping to Plato's dictum to use only circular motion. In his scheme each of the heavenly bodies was placed on the inner sphere of a set of two or more interconnected spheres, all centred on the earth, whose simultaneous rotation about different axes produced the observed motion. Although Eudoxus' plan was ingenious, in all probability he did not even try to incorporate detailed numerical data into his construction, but merely attempted to find a general geometrical – geocentric – model.

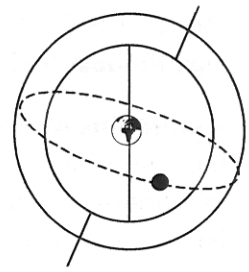


Figure 3.5

Aristarchus of Samos (ca. 310 to 230 B.C.). As reported by Archimedes, Aristarchus hypothesised "that the fixed stars and the sun remain unmoved and that the earth revolves about the sun in the circumference of a circle, the sun lying in the middle of the orbit" presenting a *heliocentric* model. The chief objection to Aristarchus' theory was that it implied that the appearance of the fixed stars would change as one viewed them from different parts of the earth's orbit. Aristarchus met this objection by further assuming that the distance to the fixed stars was so enormous that this effect would be unnoticeable. Other astronomers at the time could not bring themselves to believe that these huge distances were possible. Conflicts between sciences and religion evidently date back to ancient times.
In a treatise from ca. 260 B.C entitled *On the Sizes and Distances of the Sun and the Moon*, Aristarchus was able to derive a reasonable approximation of the distance between the earth and the moon.
If the plate on which the shadow falls in a sundial is hollowed out to form a hemisphere, then it can easily be arranged to indicate the angular position of the sun: Aristarchus invented such a sundial, called a *scaphe*.

Eratosthenes of Cyrene (ca. 280 to 200 B.C.), mathematician and chief librarian at the museum at Alexandria, a post he was to hold for the last 40 years of his life. His three-volume *Geografikè* was the first scientific attempt to put geographical studies on a sound mathematical basis. This work was the most accurate map of the world that had yet appeared and the first to use a grid of meridians of longitude and parallels of latitude. Today Eratosthenes is best remembered having devised a practical method for calculating the earth's circumference in his treatise *On the Measurement of the Earth.* The extraordinary thing about Eratosthenes' achievement is its simplicity. Eratosthenes observed that at noon during the summer solstice (~ 21 June) on the tropic of cancer, the sun cast no shadow in the Egyptian town anciently called Syene, today known as Aswan. (Story has it that Eratosthenes confirmed the position of the tropic by observing the water in a deep well). At the same hour in Alexandria, which he took to lie on the same meridian as Syene, the shadow of the sun cast in a *scaphe* showed an angle of $1/25$ of the hemisphere or $1/50$ of the full circle. Assuming the sun's rays to be parallel at Alexandria and Syene, as shown in Figure 3.6, the earth's circum-ference is given by 50 times the distance from Alexandria to Syene. This was estimated to be5,000 *stades*, 40 *stades* were supposed equal to the Egyptian σχοινος, and the σχοινος was 12,000 royal *cubits* of 0.525 meters each, which all in all yielded 6.3 km for the length of 40 *stades* and about 39,375 km for the circumference of the earth (ca.25,000 miles). Real circumference of

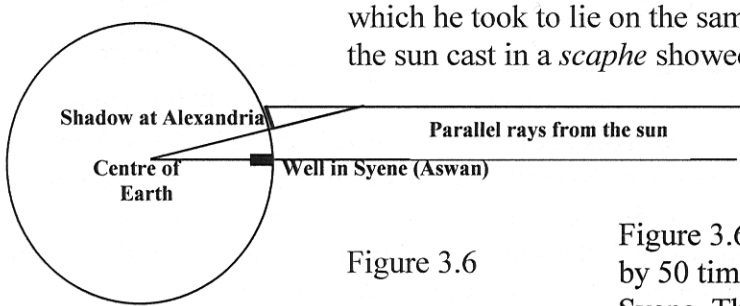


Figure 3.6

the earth is 40,007 km. Later accounts placed the figure at 252,000 stades, possibly in order to lead to the round figure of 700 stades per degree.

Apollonius of Perga (ca. 260 to 200 B.C.), was a celebrated astronomer; the favourite mathematical device in antiquity for the representation of the motions of the planets is apparently due to him. Whereas Eudoxus had used concentric spheres, Apollonius proposed instead two alternative systems. According to the Pythagorean prejudice for the beauty and perfection of the circle, the motion of the sun and planets had to be circular. However, their deviation from circular orbits was great enough to have been observed and required explanation. To reduce celestial motion to combinations of circular movements, Apollonius had worked out one system made up of *epicyclical* motions and the other involving *eccentric* motions. In the first system, an ingenious scheme of epicycles, or small circles, (orbits of planets), having their centres on the circumference of other circles. In the epicycle system, each planet travels around the earth in a large circle, called a *deferent*; this circle does not represent the true path of the planet, but rather the path of the centre of a small circle, the epicycle, around which the planet revolves. In fact, it was the necessity for introducing numerical parameters into these geometrical models, which led to the invention of trigonometry.

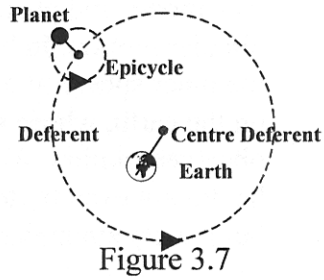


Figure 3.7

The Babylonians, sometimes before 300 B.C., initiated the division of the circumference of the circle into 360 parts, called degrees and after that the sexagesimal division of these degrees into minutes and seconds. (Our words 'minutes' and 'seconds' go back to the Latin forms *minutiae primae, minutiae seconae*, meaning first fractions, second fractions, and so on).

Hipparchus of Nicaea (ca. 190 to 125 B.C.) introduced these Babylonian fractions into Greek astronomy. The chief contributions attributed to Hipparchus in astronomy were his organisation of the empirical data derived from the Babylonians, the drawing up of a star catalogue, the development of a table of trigonometry, and finally, the discovery of the *precession* of the equinoxes.

Viewed from top (North Pole) the earth is rotating uniformly counter-clockwise once each 24 hours and makes its annual circuit of its solar orbit also counter-clockwise. The axis of the rotating earth forms an angle with the plane of its solar orbit of 23°27'. (The same angle between the equator and ecliptic). The equator and ecliptic intersect at two diametrically opposite points, the vernal and autumnal equinoxes (when day and night are of the same length March 21 and September 23). During the year, the sun rises in 12 different signs of the zodiac while its axis points to the pole of the ecliptic.

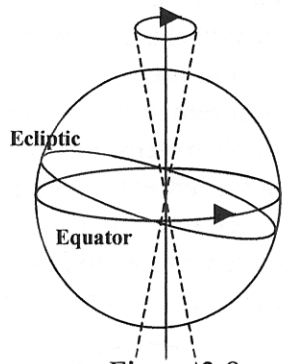


Figure 3.8

The earth, like a (humming)top, wobbles as it spins. The wobbling motion of its axis traces a circular cone in space (clock-wise), see Figure 3.8. Nearly 25,800 years pass before the earth's axis completes one circle. This course of time is called the Platonic Year. A consequence of this wobbling *precession* of the earth's axis is a continual slow change in the annual appearance of the heavens of ca. 50" per year or 1° per 70 year. This gives rise to an equally slow change in the position at which the sun appears to rise each year on the vernal equinox relative to the zodiacal constellations: the *precession of the equinoxes*. The order of zodiacal signs of the constellations in which the vernal equinoctial sun rises as the earth's axis precesses is in reverse sequence. This nearly 25,800-year period of the great rotation of the heavens was first determined by Hipparchus.

Claudius Ptolemy of Alexandria[4)] (ca. 100 to ca 160 A.D.). His *Geography* was a compilation of places in the known world along with their latitudes and longitudes, which included a discussion of the projections needed for map making. Ptolemy is most famous today for his *Mathematical Syntaxis* 'The Mathematical system', a work in thirteen books, which contained a complete mathematical description of the Greek model of the universe. Many centuries after it was written, Islamic scientists began calling it *al-Magiste* 'the Greatest' to distinguish it from lesser astronomical works. Ever since the book has been known as the *Almagest*. In this book, partially derived from Apollonius, he refined the system of epicycles to an admirable degree of accordance with the astronomical observations that it was destined to remain the supreme authority on astronomy until the publication of Copernicus' *De Revolutionibus* (1543), see 8.20.2.

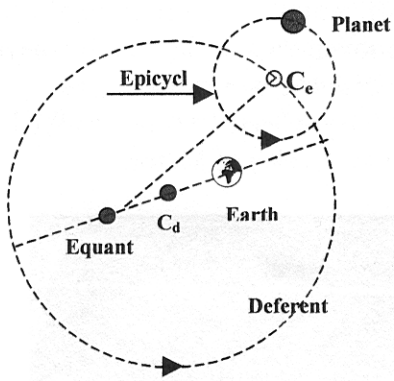


Figure 3.9

In Ptolemy's system the planets move on the epicycle, whose centre C_e moves uniformly counter-clockwise on the deferent with centre C_d.. The earth is outside the centre C_d of the deferent; the distance C_d to the earth is called the eccentricity e of the deferent. Ptolemy introduced a new geometric point, the Equant E such that E to C_d also equals e; the line through E, C_d and the earth is fixed relative to the stars. Seen from the Equant the motion of C_e is uniformly.

Central to the calculations of Ptolemy's chords (sine table) was a geometric proposition still known as 'Ptolemy's theorem': If ABCD is a (convex) quadrilateral inscribed in a circle, than AB x CD + BC x DA = AC x BD; that is: the sum of the products of the opposite sides of a cyclic quadrilateral is equal to the product of the diagonals. See Figure 3.10

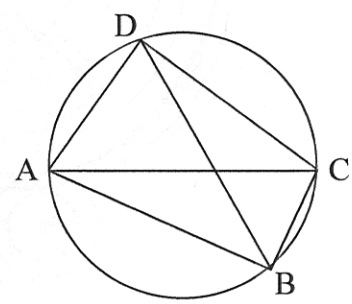


Figure 3.10

Ptolemy used the sign o, the first letter of the Greek ου δε ν, meaning ' nothing', sign for a missing factor in his Babylonian sexagesimal fractions e.g.:

μα ο ιη = 41^0 00' 18"

or ο λγ δ = 0^0 33' 04"

3.8 Antikíthera / Derek de Solla Price [5)]

The earliest surviving example of an astrolabe dates from 1062 A.D., but the writings of Ptolemy imply that the techniques were known by Hipparchus of Nicaea. The next major step in creating an analogue device is to incorporate some form of drive mechanism. It used to be thought that sophisticated gearing, involving such items as differential gears did not occur until about the middle 1500s.

In 1900 a group of Greek sponge fishermen found in the Aegean See, near the small island of Antikíthera a wreck of a large Greek ship. It appears to have been on its way (ca. 70 B.C.) from Rhodes to Rome when it was caught in a storm and sunk. Among statues and artefacts, the Greeks raised a large lump of encrusted bronze that appeared to be some collection of gears that had once been inside a wooden box, about 1ft high, 8 in wide and 4 in deep, with a crank extending from the side. The front and back of the case appear to have had dials, which were meant to represent the position of the sun, moon, and the planets. The Roman philosopher Cicero, when on a visit to Rhodes, reported seeing a device which reproduced the motions of

the sun, moon, and the five visible planets (and he was right!). In 1972 the device caught the interest of professor Derek de Solla Price of Yale, expert on ancient clockwork mechanisms. He arranged a special cleaning and X-ray photographs of the interior, to reveal the sections that were impossible to see in any other way. He reconstructed within 15 years the device with 32 gears and a sophisticated set of gears called an epicyclical differential. This differential was needed to bring in accordance the phases of the moon with the movements of the sun. It was a nautic instrument to determine the location on see based on a geocentric world-view. See Figure 3.11 and Figure 3.12 The device shows that when the sun have been 19 times walked through the zodiac, the moon already did it 235 times according the cycles of Meton.

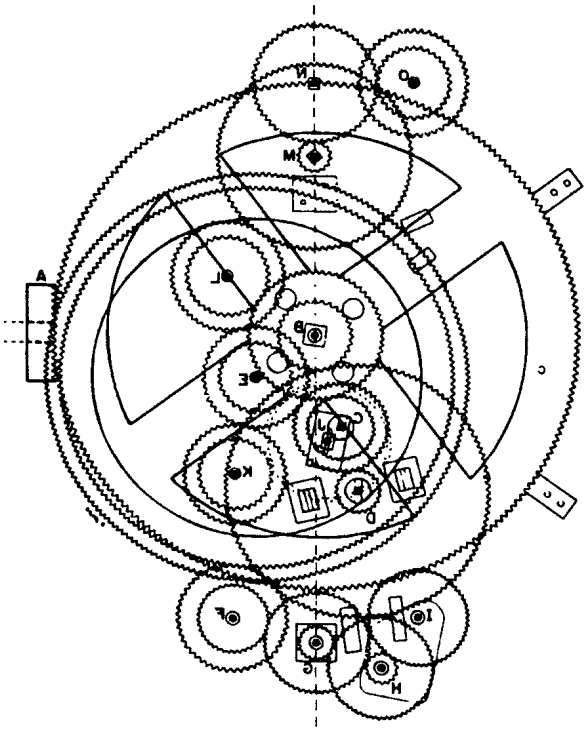


Figure 3.11

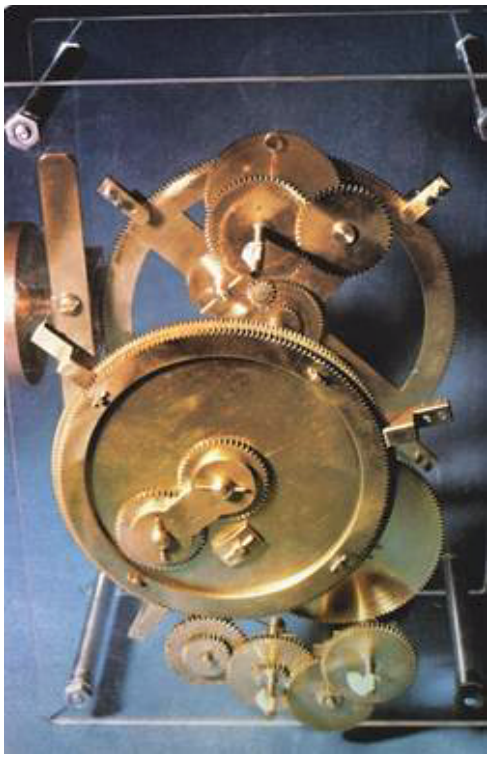


Figure 3.12

1) J. M. Pullan
The History of the Abacus
With a list from Mabel Lang containing 12 Greek abacus tables

2) David M. Burton
Burton's History of Mathematics, An Introduction

3) Victor J. Katz
A History of Mathematics

4) Boyer & Merzbach
A history of Mathematics

5) Michael R. Williams
A History of Computing Technology

Chapter four, *The Romans*

4.1 The Roman numerals

The Roman Empire existed for a period of ca. 1000 years – from the constitution of the Republic in 510 B.C. to the fall of the western part 476 A.D. All this time the Romans continued to make use of an archaic system for representing their numerals. From the Roman numerals the one I, the five V, and the ten X, already were in use in antiquity on tallies. A similar primitive system is in common use today for counting a number of objects; strokes are grouped in five's like this: 卌. On the same way by the Romans the unit I was crossed out to get a decade: ⋊ or ⋉, the symbol for thirty 卌, which is found confirms this view; hundred = 10 x 10 turned into Ж. The symbol for five was obtained by halving the symbol for ten: the Romans used the upper half of X =V, the Etruscans used the lower half Λ, (on an Etruscan coin from the 5th century B.C.). The Etruscans used for fifty the lower half of hundred ↑ , Romans used, again, the upper half of hundred ↓ ; this developed via ⊥ and ⊥ into L.
A milestone and signboard from 130 B.C. giving distances, among others, to:
MVRANUM ⊥XXIIII = 74 and VALENTIAM C⊥XXX = 180.
The Etruscans composed the sign for 100 Ж to)|(,
from this the Romans kept the right part (, this is also the initial letter of centum = hundred.
An ancient numeral for one thousand in Latin inscriptions resembles a Greek letter phi φ. This developed to (|). Later |) and than D, the right-hand side of 1000 = 500.
On the same above mentioned stone the number DCCCCXVII for 917 : the sign D for 500.
From thousand this (|) was extended to ((|)) for ten thousand and (((|))) for hundred thousand. See Figure 4.1.
A bar above a given number occasionally indicated thousands: $\overline{\text{VII}}$CCCL for 7350 ,
a double bar indicated multiplication by 1000^2 : $\overline{\overline{\text{XV}}}$ = 15,000,000. From the time of Hadrian, first century A.D., inscriptions are found with a bar above and vertical bars at the side which indicate hundred thousands times a given number. Thus $\overline{\text{X}}$ is ten times hundred thousand (decies centana milia) = one million. Also $\overline{\text{C}}$ = 100 x hundred thousand.
The use of IV for 4 or IX for 9 can be found as early as 130 A.D., but these symbols did not become popular until about 1600 A.D., when Roman numerals were no longer so important in everyday use. Only as from the Middle Ages M was used for thousand, the initial letter of mille.

There is a Roman illustration in the form of an engraved seal, now in the Cabinet des Médailles of the Bibliothèque National, Paris. It shows a 'calculator' in exact the same situation as the receiver of attribute on the Darius Vase. The writing tablet in his left hand shows the signs for thousands, hundreds, tens and units ΦΘ × I while he is placing the calculi on the table abacus with his right hand.

Calculating with these numerals is very difficult and tedious. But how did the Romans perform their calculations? They used a counting board. Of this there were two different shapes: a board with pebbles, comparable with the Greek Salamis table, and a hand-abacus.

4.2 The Roman hand−abacus

A small number of remarkable 'bead-slot computers' or 'pocket calculators' of Roman origin has survived. They are small enough (about 90 x 125 mm) to be held in one hand while the beads are moved in the slots with the other hand, the normal Roman method of calculation.

One of them is in the British Museum, another is in the Cabinet des Médailles of the Bibliothèque Nationale in Paris, and a third is in the Museo Nazionale Romano, see Figure 4.1

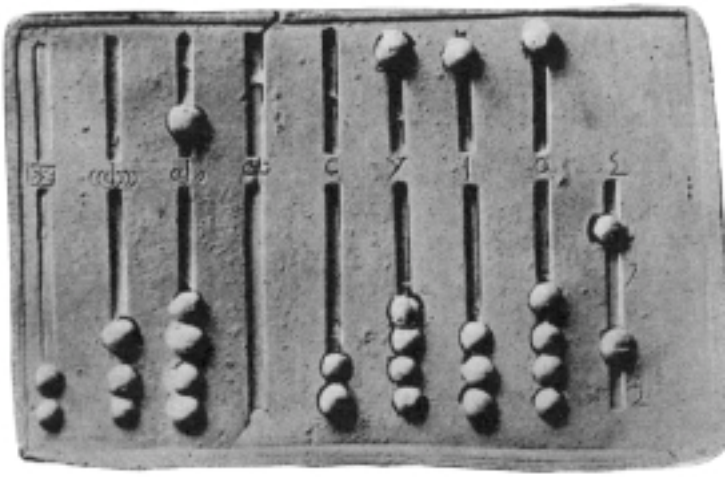


Figure 4.1

The Roman hand-abacus consists of an oblong piece of bronze, provided with slots. In these slots beads can be pushed up and down, four beads in the lower slots, one bead in the upper slots. Each bead in the upper slots has the value of 5 times of the lower beads; 6 times for those of the twelfth parts (θ = uncia). So the abacus of Figure 4.1 is not anymore complete.

Between the slots the symbols are indicated for, from the left to the right:

$\boxed{X}$ (((|))) ((|)) (|) (X |

The millions, hundred thousands, ten thousands, thousands, hundreds, tens, unities (1 = as),

To the right is another separate slot for the fractions of the uncia:
½ = semunsia = ƐƐ ¼ = sicilicus = ..) ⅓ = duella = 2

4.3 The Roman unity for coins

The as was one of several Romans unities for coins equivalent to the weight of a pound of copper. It was subdivided according to:

1 as = 12 uncia. On the abacus 2^{e} from the right, indicated with θ
½ as = 6 uncia = 1 semias
1 uncia = 24 scruple (scruple = small pebble)
½ uncia = 12 scruple = semuncia = duella sicilicus = ƐƐ
¼ uncia = 6 scruple = sicilicus (sickle) = ..)
⅓ uncia = 8 scruple = duo-sextuale = duella = 2
½ scruple = 4 calcus = 1 obolus
¼ scruple = 2 calcus = 1 ceratus (= 1carat)
⅛ scruple = 1 calcus

With this hand-abacus only simple calculations could be performed. For addition or subtraction the beads of the first number were ‘inserted’ by pushing up or down the beads, the next number mentally added or subtracted, the result again ‘inserted’ and so on.

4.4 Multiplication on the Roman counting board with loose counters [1)]

Although Roman abacus calculi are frequently found, no example has yet appeared of a roman table abacus.

			③ ②		⑦ ⑥	
$\boxed{X}$	(((|)))	((|)	(|)	(	x	|
		② ② ② ③	③ ③ ③ ④ ④ ④ ④	④ ④ ⑤ ⑤ ⑤	⑥ ⑥ ⑦ ⑧	⑧ ⑧ ⑧ ⑧ ○
	37		○	○	○ ○ ○	○ ○
	5612			○	○	○ ○

Performing a multiplication or a division on the Roman counting board with loose counters looks easily. The most important rule to perform a multiplication goes: the column in which the unity of a product is placed is one more to the right than the sum of the columns of both factors.
Example: multiply 5612 by 37.
First the two factors are formed at the bottom.
Start with the left-hand factors 5 x 3 = 15: put counters in column 4 + 2 - 1 = 5 (+6) with ;
5 x 7 = 35, in column 4 + 1 - 1 = 4 (+5) with ②
Now the counter for 5000 can be removed;
6 x 3 = 18 in column 3 + 2 - 1 = 4 (+5) with ③6 x 7 = 42 in column 3 + 1 - 1 = 3 (+4) with ④
Now the counter for 600 can be removed;
1 x 3 = 3 in column 2 + 2 - 1 = 3 with ⑤
1 x 7 = 7 in column 2 + 1 - 1 = 2 with ⑥ and
2 x 3 = 6 in column 1 + 2 - 1 = 2 with ⑦
2 x 7 =14 in column 1 + 1 - 1 = 1 with ⑧.

Now every 5 counters in each column in the lower section must be replaced by one in the same column in the upper section. Every 2 counters in each column in the upper section must be replaced by one in the lower section of the next column.

			○	○		
$\boxed{X}$	(((|)))	((|)	(|)	(	x	|
	○ ○ 2	0	○ ○ 7	○ 6	○ ○ ○ ○ 4	○ ○ ○ ○ 4

Result: 207644 or $\overline{CC}\ \overline{VII}$ DCC XL IV

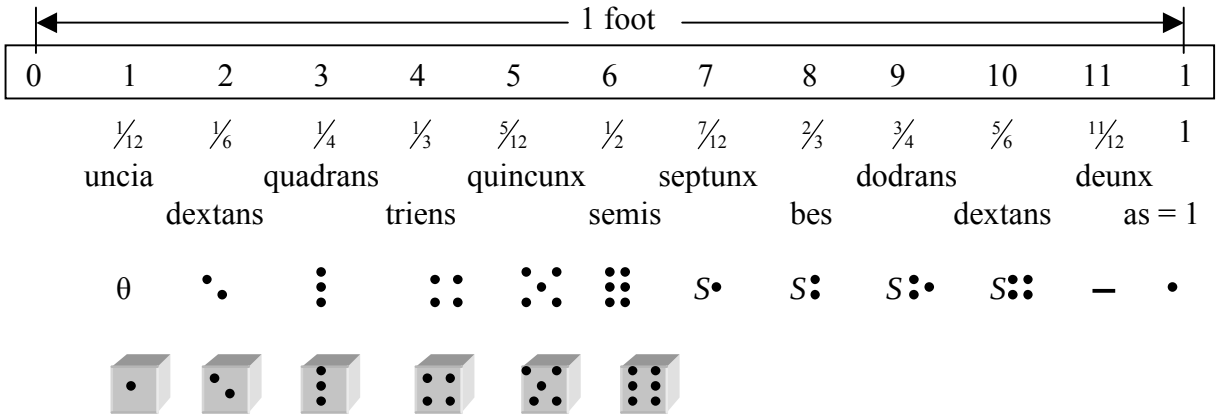
There was no symbol for an omitted column, no 'zero'; the appropriate column was symply left empty. The zero is superfuous as well on the board as in writing.

Some denominations:

the Latin calx = limestone,
calculi = (calculating)pebbles, (origin of the words calculation and calculator)
calculos poneren = lay pebbles (add),
calculos subducere = remove pebbles (subtract),
calculones = calculating teachers (mostly slaves),
calculatores = free calculating teachers,
arenarii = mathematicians working on sand covered tables.

4.5 The Roman units for measures and weights [2)]

The Romans were very practically; theory had not their interest. In their Empire they introduced the duodecimal system for coins, measures and weights. The unit of this system, especially used by technicians, was the as = 1 Roman pound copper, (pound = libra, shorted by £, the symbol for the British pound sterling);
The twelfth part of the pound was the uncia (from which we get the words ounce - for 100 gram - and inch). One did not dount 1, 2, 3 uncia, but each twelfth part of the as had its own name and symbol:



symbols which still exist on dice, dominoes and playing cards (as = ace).

The unit for length was the foot (296 mm). The twelfth part, the uncia (inch) was again divided in 12 parts: ½ inch = semuncia, ⅓ inch = duella etc. One inch was 24,8 mm (the English inch is 25,4 mm, difference 0,6 mm).

In stead of the inch the Roman technicians used also a smaller unit, the finger (digitus). The finger was 1/16 part of a foot. Mostly fingers were not used, but for six fingers one took ⅜ foot, for 12 fingers ¾ foot etc. This is exactly the same sub division which is still used in the Imperial Unit System for screw thread and gas (central heating) pipes!

The Roman foot, called a *pes*, with plural *pedes*. Five feet was called *passus* or *pace* = one step. Thousand steps (paces) = *milia pasuum* = one Roman mile. A surveyor with a very even step size measured off this mile.

The cubic foot, called an *amphora*, was the fundamental unit of liquid measure. ⅓ amphora was called a *modius*, the unit of dry measure.

1) as supposed by: Karl Menninger in: Number Words and Number Symbols

2) in an article from George A. Ritzler:
Fritz Kretzschmer Technik und Handwerk im Imperium Romanum

Chapter five, *Latin Europe to ca. 1000 A.D.*

We use the term Latin Europe to denote those parts of Europe in which Latin was the principal language employed by the literate for communication. It comprised chiefly Italy, Christian Spain, France, the Low Countries, England, and Scandinavia.

5.1 The end of the Roman Empire in the West

The end of the Roman Empire in the West usually is set at 476, for in this year Odoacer, a Goth, displaced the incumbent Roman Emperor Romulus Augustus.
After the collapse of the Roman Empire in the West under the onslaught of various 'barbarian' tribes, in the East the authority was continued by the Emperor of Constantinople, while the pope in Rome, respectively the Catholic Church, tried to continue as best they might the cultural tradition of the Roman Empire. For the next five centuries, however, the general level of culture in Europe was very low. Serfs worked the land and few of the barons could read or write, let alone understand mathematics. In fact, there was little practical need for the subject, because the feudal estates were relatively self-sufficient. From the sixth to the ninth century a considerable amount of classical learning was preserved in the monasteries in Ireland, which had been spared some of the tumult that accompanied the decline of Roman power in the rest of Europe. For the study of the 'science of numbers', however, virtually all early medieval Europe had available were Latin versions of Nicomachus' *Arithmetic* by Boethius (see 5.2). In the church and monastery schools of the Middle Ages arithmetic was included largely for the computation of Easter; the technical treatise on this topic was called a *computus*, (see 5.3). This important yearly returning problem had to be solved to prevent the coincide of the Jewish Pascha and the Catholic Easter. During the Synod of Nicaea in 325 A.D. it was fixed how to compute the correct date for Easter: the Jewish Pascha is on the eve of the first full moon in spring and the Christian Easter is on the first Sunday after the first full moon on or after the equinoctial vernum, the moment in spring upon which day and night are of the same length; in the autumn this is called the equinoctial autumnal. The calculations were performed with pebbles on a counting board and/or with the fingers, results were written in Roman numerals. How complicated these calculations were, is extensively described in *Computus Magistri Jacobi*, a schoolbook for *computus* from 1436. **1)**
From this period a few names are known or worth to be mentioned:

5.2 Boethius

Anicius Manlius Severinus Boethius, (*ca. 480 Rome, † 524), philosopher, mathematician, statesman and author, died as a martyr for his faith. He was the author of textbooks for each of the four mathematical branches of the seven *artes liberalis*:
- an *Arithmetica* that was only an abridgement of the *Introductio* of Nicomachus of Gerasa (~100 A.D.)
- a *Geometry* based on Euclid and including statements only, without proof, of some of the simpler portions of the first four books of the *Elements*;
- an *Astronomy* derived from Ptolemy's *Mathematical Syntaxis*;
- a *Music* that is indebted to the earlier works of Euclid, Nicomachus, and Ptolemy.

He watered down the harder parts of the classical Greek works of mathematics to adapt them to the intellectually degenerate time in which he lived.
In prison, before he was executed, he wrote his most celebrated work, *De consolatione philosophiae*.

5.3 Bede

The Christian liturgy made new demands on time reckoning. The Julian calendar maintained its position for civil purposes. It was also used for defining ‘fixed’ ecclesiastical events such as Christmas, Epiphany and the Annunciation. On the other hand, the ‘movable’ feasts of Easter and the derived dates of Pentecost and the beginning of Lent were determined by rules going back to the lunar calendar of ancient Israel. Thus Easter Sunday had to be the first Sunday on or after the first full moon after the equinoctial vernum.
The determination of this date in the civil calendar was a rather difficult problem, which found no general solution until the beginning of the sixth century. All these questions were exhaustively dealt with in two manuals about computus:

Liber de temporibus and *De temporum ratione* – On the Theory of Time-reckoning

written in 725 by the Benedictine monk Bede of Yarrow (672 – 735), the most famous teacher of the early Anglo-Saxons. The manuals have an introduction:

De computo vel loquela digitorum – About counting and speaking with the fingers.

This introduction was very long in force as the manual for arithmetic in Europe. Bede gives a description of upwards of fifty finger symbols, the numbers extending through one million, see Chapter 16.
Unsurpassed in clarity and comprehensiveness it remained for centuries the most impressive monument of monastic science. It was also Bede who first made consistent use of the Christian era in his History of the English Church and People, in which the years are numbered from the birth of Christ.

5.4 Alcuin of York and Charlemagne

Alcuin of York (*ca. 730 York, † 804 Tours), studied at a monastery school at York. On Charlemagne’s request he settled in 782 in the Franconian Empire and became leader of the court school at Aix-la-Chapelle. Besides his principal work:

De fide sanctae et individuae Trinitatis – about the Trinity

to him also a collection of fifty-three arithmetical problems is attributed, entitled

Propositiones ad acuendos iuvenes – Propositions for Sharpening Youths.

The problems often require some ingenuity for solving, but do not depend on any mathematical theory or rules of procedure; among others: "A wolf, a goat and a cabbage have to take across a river. Besides the ferryman the ferry only can take one of these three on a passage. How should the ferryman handle to get beyond the river, without the cabbage is eaten by the goat or the goat is eaten by the wolf?" Charlemagne recommended that the mathematics necessary for Easter computation be part of the curriculum in church schools.
Charlemagne’s attempt to promote the *artes liberalis*, however, encountered great obstacles, as his empire was divided among his three sons after his death. In addition the ninth and the tenth centuries saw the last waves of invaders from the north – the Vikings, who disrupted commerce and civilisation both on the continent and in Britain and Ireland until they themselves became Christians and adopted a settled way of life.
Nevertheless, Charlemagne’s directive to create cathedral and monastery schools had a permanent effect.

5.5 Gerbert d'Aurillac

A career in public life for one not of noble birth was necessarily an ecclesiastical career. Born to lower-class but free parents in Aquitanië (Auvergne), Gerbert of Aurillac (*ca. 940 – 1003), was one of them who benefited from Charlemagne's decree that monasteries and cathedrals must have schools. He was educated in Latin grammar at the monastery of Saint-Gerard in Aurillac, and excelled in mathematics and astronomy. In 967 he left the monastery in order to accompany a Count Borel of Barcelona on his journey home. His stay in the Spanish border country, where he was able to continue his studies as a result of his friendship with Bishop Hatto, greatly enriched Gerbert's mathematical knowledge.
(The Arabs had been in Spain since 713 (see Chapter 7). In 732 the Arabs were beaten in France near Poitiers by Charles Martel and drove back into Spain).
However, these western Arabs used the so-called gubar figures for their computations, it is more than likely that Gerbert became acquainted with those figures there. There was no zero sign – those columns on the abacus that had no digits were simply left empty. The zero began to play its important role only in written computations.
Back in France, between 972 and 982, he studied dialectics and taught his 'own' mathematics at the cathedral school in Rheims. Since Gerbert and his disciples had no knowledge of written computations, they did not grasp the essence of the 'new' systems of numerals. They merely adopted the symbols, whose mysterious origin and significance were beclouded by their obscure and partly Arabic names.
For a while Gerbert was an abbot in Ravenna. In 991, he returned to Rheims, where he was elevated to Archbishop. In 999 Gerbert became Pope Sylvester II. In May of the year 1003 Gerbert died at the age of 63.
Gerbert wrote *Regulae de numerorum abaci rationibus* – Rules for Computation with numbers on the Abacus.

5.6 The counting board of Gerbert.

From one of Gerbert's students we know exactly what Gerbert's, and hence, the early Medieval counting board looked like. His monastic abacus had 27 parallel columns and 3 more for fractions. The columns sometimes were closed off at the top by an 'arch'. This was called *Arcus Pythagorei*, 'arch of Pythagoras' because in the Middle Ages this Greek was erroneously believed to be the inventor of the abacus. The columns were indicated with Roman numerals. However, the Roman 5-grouping has disappeared. In its place Gerbert for the first time used, instead of the seven pebbles that once had to be lined up in a row, a single counter marked with the appropriate number symbol for seven: ❼.

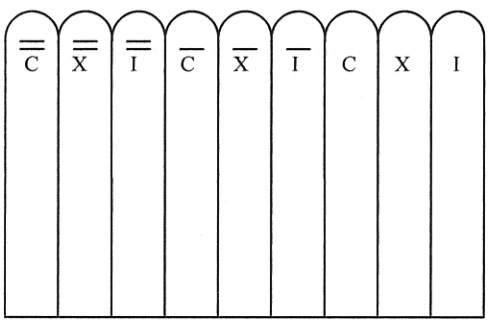


Figure 5.1

But for this purpose he did not use the Roman numerals, but *novem numero notas omnem numerum significantes disposuit*, or as a pupil of Gerbert's said he used nine symbols, with which he was able to express every number.
The counters marked with numerical symbols were called *apices* (from the Latin apex). Gerbert had some 1000 such counters carved out of horn. This was his innovation: the replacement of seven *calculi* by a single counter or apex marked with the symbol sign for 7.

The rotation, or different orientations, of the individual number symbols and apices may be due to the fact that the counters were customarily placed on the counting board in a particular manner in one monastery and differently in another.

First, let us examine two methods of division on the early medieval counting board with apices, whose procedure was described by Gerbert.

5.7 The iron division Divisio ferrea [2)]

In his *Regulae de numerorum abaci rationibus* Gerbert described among other things the iron division, the *divisio ferrea*, for his counting-board. This division was so named because this division, one of the procedures using supplementary numbers of which the Middle Ages was so fond, was so extraordinarily difficult that its hardness surpassed that of iron, as one medieval manuscript puts it.

As always in computations on the counting board, the rules of place value are extremely important. The most important rule for the products of multiplication essentially goes back to Archimedes and refers to Greek written computations, but which originated from computations on the reckoning board.
The number of ranks p of a product is equal to one less than the sum of the ranks $a + b$ of the factors; to state this in terms of an equation: $p = a + b - 1$.
Example: 20 x 30 = 600; $a = b = 2$; $p = 2 + 2 - 1 = 3$; thus, the product has tree digits.

Example: 600 divided by 20 gives 30 – tree counters, but in which column do they belong? Since 30 (a) x 20 (b) = 600 (p), we can derive the place-value for division $a = p - b + 1$, in which $p = 3$ would be the column number of the dividend 600 and $b = 2$ that of the divisor 20. The quotient 3 goes into column $a = 3 - 2 + 1 = 2$, and thus into the tens column.

Now an *iron division* as performed by Gerbert:
(We shall use the abbreviations U for units, T for tens, H for hundreds and Th for thousands.)
Example: 5612 divided by 37: first the divisor 37 is increased by the supplement $s = 3$ to the next complete decade, the level $r = 40$; (37 + 3 = 40)
Procedure: divide through the rank 40 = 4T, see line 1', 7Th ÷ 4T = 1H;
(the quotient ❶ is placed below, in column 4 - 2 + 1 = 3 of the iron division, see next page); now equate the'error' with the balance (3 x 100) (see line 1''), etc.

This results in Hindu-Arabic numerals:

	5612 ÷ 40 (37 + 3) = 100	
1' – 40 x 100	4000	(1)
	1612	
1'' + 3 x 100	300	
	1912 ÷ 40 =	40
2' – 40 x 40	1600	(2)
	312	
2''+ 3 x 40	120	
	432 ÷ 40 =	10
3' – 40 x 10	400	(3)
	32 (to transfer)	

(transferred)	32	
3'' + 3 x 10	30	
	62 ÷ 40 =	1
4' – 40 x 1	40	(4)
	22	
4'' + 3 x 1	3	
	25 is the remainder	

Quotient (1) + (2) + (3) = (4) = 151

Now on Gerbert's counting-board with apices (there is no zero apex!):

	Th	H	T	U	
					Th = thousands, H = hundreds, T = tens and U = units
divisor			❸	❼	
supplement (*s*)				❸	
rank (*r*)			❹		
dividend (*p*)	❺	❻	❶	❷	
1'(-100x40=)	❹				100 is the first part of the quotient and is placed below
	❶	❻	❶	❷	in column 4 - 2 + 1 = 3;
1''(+100x3)		❸			
	❶	❾	❶	❷	
2'(-40x40=)	❶	❻			40 is the second part of the quotient, that is
		❸	❶	❷	
2''(+40x3)		❶	❷		
		❹	❸	❷	
3'(-10x40)		❹			supplemented with 10 (40 + 10 = 50)
			❸	❷	and is placed below in column 3 - 2 + 1 = 2
3''(+10x3)			❸		and finally
			❻	❷	
4'(-1x40)			❹		1 as third part of the quotient;
			❷	❷	
4''(+1x3)				❸	
			❷	❺	this is the remainder, 25;
		❶	❹	❶	
			❶		
		❶	❺	❶	quotient, total 151

5.8 The golden division Divisio aurea [2)]

The golden division uses the divisor directly, as we do; it is, in fact, our customary method of division. The divisor always appears on the same level as the highest digit of the dividend, and the rank of the quotient is ascertained by the afore mentioned rule.
Again, to divide 5612 by 37:

Step 1

Th	H	T	U
3	7		
5	6	1	2
~~5~~	6	1	2
2			
	1		

Th = thousands, H = hundreds,
T = tens and U = units
5Th ÷ 3 = 1, first figure of the quotient;
1 x 3Th = 3Th, remainder (5-3=)2Th;
cancel 5, set 2;
first figure of the quotient.

Step 1'

Th	H	T	U	
3	7			
5	6	1	2	1 x 7H = 7H
~~2~~	~~6~~	1	2	rem. = 19H
1	9			cancel 26,
	1			set 19

Step 2

Th	H	T	U
	3	7	
5	6	1	2
~~1~~	~~9~~	1	2
1	5		
	4		
	1	5	

divisor 1 position to the right;
19 ÷ 3(7) = 5, second figure
of the quotient;
5 x 3H = 15H rest 4H;
cancel 19, set 4;
two figures of the quotient.

Step 2'

Th	H	T	U	
	3	7		
5	6	1	2	5 x 7T = 35T
	~~4~~	~~1~~	2	remainder = 6T
	3	5		cancel 41;
		6		set 6.
	1	5		

Step 3

Th	H	T	U
		3	7
5	6	1	2
		~~6~~	2
		3	
		3	
	1	5	1

divisor another pos. to the right;
6 ÷ 3(7) = 1, third figure of
the quotient;
1 x 3T = 3T remainder 3T
cancel 6, set 3;
three figures of the quotient.

Step 3'

Th	H	T	U	
		3	7	
5	6	1	2	1 x 7 = 7
		~~3~~	~~2~~	remainder = 25
			7	cancel 32;
		2	5	set 25;
	1	5	1	quotient.

Below the 'Arabic' numerals, as Gerbert possibly used on his apices.
The pictures and names of the numerals are from a 14th century manuscript of a monastery.

celentis	temenias	zenis	caltis	quimas	arbas	ormiss	andras	igin	sipos
1	2	3	4	5	6	7	8	9	0

1) A book completely dedicated to computus:
Marijke Gumbert-Hepp: Computus Magistri Jacobi

2) Karl Menninger
Number Words and Number Symbols

Chapter six, *India*

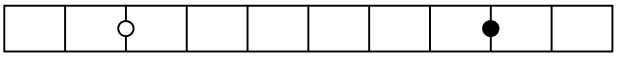
6.1 Harappa and Mohenj o-Daro

Between 1921 and 1923 a series of archaeological excavations along the banks of the Indus uncovered the remains of two urban centres, at Harappa and Mohenjo-Daro, dating back to about 3000 B.C. Similar remains have been found spread across the Indus Valley, parts of East Punjab and Uttar Pradesh. This so-called Harappa culture was a highly organised society. There is every possibility that they were skilled in mensuration and practical arithmetic of a kind similar to what was practised in Egypt and Mesopotamia. Alas, the Harappa script remains undeciphered. Archaeological finds do provide some indications such as:

1) A number of different plumb-bobs of uniform size and weight, showing little change over five hundred years. The weights could be classified as 'decimal': if we take the plumb-bob weighing approximately 27.584 grams as a standard, representing 1, the other weights form a series with values of:

0,05 0,1 0,2 0,5 1 2 5 10 20 50 100 200 and 500.

2) Scales and instruments for measuring have been discovered at Mohenjo-Daro and Harappa. The 'scales' are 66,2 mm long, with nine carefully sawn, equally spaced parallel lines, on an average 6,7056 mm apart. One of the lines is marked by a hollow circle, and the sixth line from the circle is indicated by a large circular dot. The distance between the two markers is 33,5 mm, (1,32 inches) and has been named the 'Indus-inch'. The Sumerian *shushi* is exactly half an 'Indus-inch'.



6.2 Vedas, Vedangas en Sulbasutras

Around 1500 B.C. a group of people descended from the North and destroyed the Harappa culture, but not before they had absorbed some of its features. These invaders are often refered to as *Aryans*. The Aryans were a pastoral people, speaking a language, which belonged to the Indo-European family.
Over the years this language, *Sanskrit*, developed sufficiently to become a suitable medium for religious and scientific discourse. In Sanskrit much was passed on orally, in rhyme, the so-called *Vedas*, not written down. To make it easier to memorise, since numbers have few rhymes, words were adopted for numbers. Numbers as the rhyming word wanted to, were taken from life. Zero was represented by *sunya* = void or *ambara akasa* = heavenly space, one by *rupa* = Moon or *bhumi* = Earth, two by *yama* = twice or *kara* = hands and so on.
The potential for scientific use of Sanskrit was greatly enhanced as a result of the thorough systematisation of its grammar by Panini, about 500 B.C. Sanskrit served as a useful medium for recording early scriptural texts such as the *Vedas* (lyrical collections of hymns, prayers, incantations, and sacrificial and magic formulae).
Vedic literature (~1000 to 500 B.C.), divided into several *Vedas* (*Rigveda* for hymns and prayers, *Yajurveda* for sacrificial formulae etc.)
For mathematics, the 'appendices' provided an important source to the main *Vedas*, known as the *Vedangas*. These *Vedangas* were classified into six branches of knowledge: among others metronomy, astronomy and rules for rituals and ceremonials, the *Kalpasutras.* This ritual literature included *Srantasutras*, which gave directions for constructing sacrificial fires at different times of the year. Part of this literature dealt with the measurement and construction

of sacrificial altars, and came to be known as the *Sulbasutras.* Later the word *sulba* came to refer to the cord of rope used in measuring altars. Most of what we know of *Vedic* geometry comes from these *sutras*.
The *Sulbasutras* provided instructions for two types of ritual, one for worship at home and the other for communal worship. Square and circular altars were sufficient for household rituals, while more elaborate altars whose shapes were combinations of rectangles, triangles and trapeziums were required for public worship.
The earliest of the *Sulbasutras* recorded by Baudhayana between 800 and 600 B.C., contains a general statement of the Pythagorean theorem: The rope, which is stretched across the diagonal of a square, produces an area double the size of the original square. Katyayana gave a more general proposition: The rope stretched along the length of the diagonal of a rectangle makes an area, which the vertical and the horizontal sides make together.

6.3 Application of Pythagoras' theorem [1)]

Figure 6.1 illustrates how this proposition was applied in the construction of altars. The instructions given for construction of a communal altar in Apastamba's *Sulbasutra* are, in modern notation, as follows (Apastamba from the same period as Baudhayana):
It had to be an isosceles trapezium ABCD, where AD were 24 *padas* (literally feet) and BC were 30 *padas*. The altitude of the trapezium (i.e. the distance between the midpoints X and Y of AD and BC) had to be precisely 36 *padas*.
1) With the help of a rope mark out XY, which is precisely 36 *padas*.
2) Along this line, locate points P, R and Q such that XP = 5, XR = 28 and XQ = 35 *padas*.
3) Construct the perpendiculars at X and Y.
4) Use the fact that the triangles APX, DPX, BRY and CRY are right-angled triangles with integral-valued sides to locate points A, B, C and D. In other words, make AXD 24 *padas* and BYC 30 *padas*. Join AB, BC, CD and DA.
5) As a check, join AC and BD. If the design is right, they should intersect at O on XY.

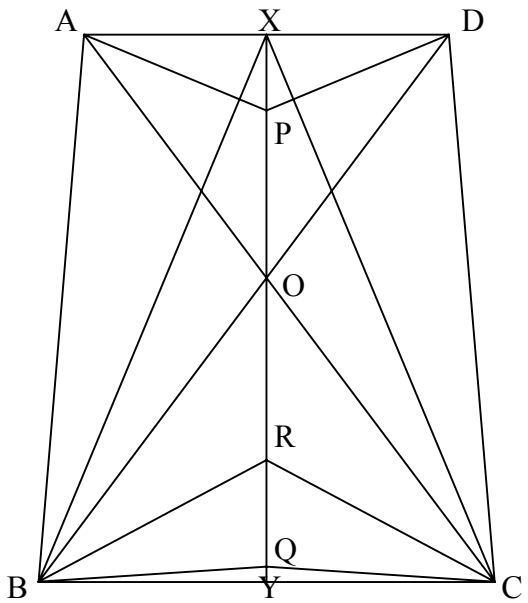


Figure 6.1

AX = XD = 12 *padas*
BY = YC = 15 "
XP = 5 "
PR = 23 "
RQ = 7 "
QY = 1 "
XY = 36 "

Implied in these directions for construction are the following right-angled triangles with integral sides:

ΔAPX and ΔDPX	with sides	5,12,13
ΔAOX and ΔDOX	with sides	12,16,20
ΔBRY and ΔCRY	with sides	8,15,17
ΔBOY and ΔCOY	with sides	15,20,25
ΔAQX and ΔDQX	with sides	12,35,37
ΔBXY and ΔCXY	with sides	15,36,39

The earliest of the *Sulbasutras* recorded between 800 and 600 B.C. by Baudhayana also contains procedures to merge two equal or unequal squares to obtain a third square. Figure 6.2 shows the construction. In modern notation, let ABCD and PQRS be two squares to be combined, and let DX be equal to SR. Join AX. The square on AX is equal to the sum of the squares ABCD and PQRS. The original explanation then points out that $DX^2 + AD^2 = AX^2 = SR^2 + AD^2$, which shows the use of the Pythagorean theorem.

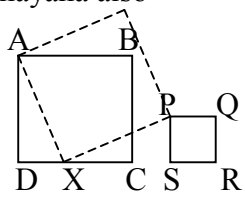


Figure 6.2

6.4 Irrational square roots: an approximation procedure [1)]

A remarkable achievement of Vedic mathematics was the discovery of a procedure for evaluating square roots to a high degree of approximation. The problem may have originally arisen from an attempt to determine the length of the sides of a square altar with an area twice as large as a square altar with sides of a standard length. An early version of Baudhayana contains an approximation procedure for obtaining $\sqrt{2}$ correct to five decimal places. Consider two squares, ABCD, each of unit side and make the construction of Figure 6.2 than the area of square EFGD is twice the square ABCD and DB = DC = $\sqrt{2}$, see Figure 6.3.

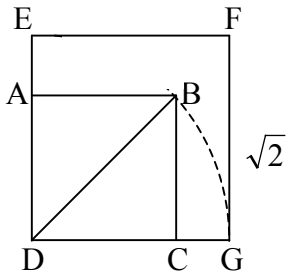


Figure 6.3

The procedure given by Baudhayana may be restated as:
"Increase the measure by its third and this third by its own fourth less the thirty-fourth part of that fourth. This is the value with a special quantity in excess."
If we take 1 unit as the dimension of the side of the square, this formula gives the approximate length of the square's diagonal as: $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} = 1{,}4142156$ ($\sqrt{2} = 1{,}414213562$)
The *Sulbasutras* contain no clue as to how this approximation was arrived at. A plausible explanation is as follows. Consider two squares ABCD and PQRS, each of unit side. PQRS is divided into three equal rectangular strips, marked 1 and 2. The third strip is subdivided into three squares, of which the first is marked 3. The remaining two squares are each divided into four equal strips, see Figure 6.4a
These 11 areas are added to the square ABCD as shown in Figure 6.4b to obtain a large square EFGD less a small square at the corner F.

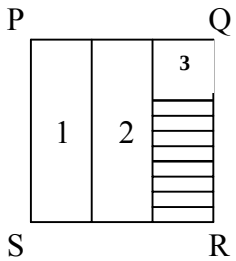


Figure 6.4a

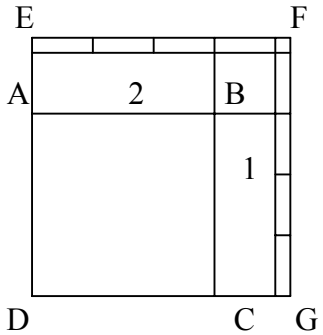


Figure 6.4b

6.5 Kharosthi numerals

The ideology of Buddha (560 – 483 B.C.) is a people's religion, which turned against that of the Brahmans. During the Buddhism came into existence a new writing and people became more interested in numerals.
When Alexander the Great on his expedition against the Persians (327 to 325 B.C.), who occupied the Northwest of India already in the 6th century B.C., on his turn occupied India, Greek and with that also Babylonian knowledge streamed into India, including arithmetic and astronomy, however with little consequences.
From the 4th century B.C. we find in India new numerals, as Kharosthi en Brahmi.

The Kharosthi numerals are found in inscriptions dating to a period from the 4th century B.C. to the 2nd century A.D. Special symbols were used to show 1, 4, and both 10 and 20. Numbers up to 100 were then built up additively:

I	II	III	X	IX	IIX	IIIX	XX	IXX			
1	2	3	4	5	6	7	8	9	10	20	50

For larger numbers the multiplication principle came into operation, with special symbols for higher powers of 10.

6.6 Brahmi numerals

The earliest trace of the more highly developed Brahmi numerals is from the 3rd century B.C. There were separate symbols for the digits 1 to 9 and the number 10 and its higher powers up to 90. For 100, 1,000 and 10,000 there is a unique symbol. The maximum number that could be represented by these symbols was 99,999

1	2	3	4	5	6	7	8	9	10	20	30	40	50	60	70	80	90

Numbers were arranged with the unit's position to the left. The number 5612, for example, could have been represented as:

2	10	6x100	5x1.000

; in words:

eka	*dvi*	*tri*	*catur*	*pañca*	*sat*	*sapta*	*asta*	*nava*	*dasa*	*sata*	*sahastra*
1	2	3	4	5	6	7	8	9	10	100	1000,

thus 5612 as: *dvi, eka dasa, sat sata, pañca sahastra*

The astronomical standard work of the Hindus, the *Sûrya Siddhânta* (System of the Sun) of the first part of the 6th century, largely founded on Greek sources, contained among others the epicycles hypothesis and a sine table given in verse form still using rhyming words for numerals; for instance 1021 was represented by moon, wings, hole, nose, thus still the units at the left.

Bhâskara I born around 520 A.D., disciple of Ãryabhata, improved the system that now became positional and got the zero, but still recited the units at the left. 5612 Is simplified to two, one, six, and five. This system of counting with words was used with the positional principle but only for the first nine units. For 301 a word had been added to indicate the zero, that became *sùnya* = empty: *eka sùnya tri*
On a donation document from the year 595 A.D. (in Indian era the year 346) that year was written with Brahmi numerals as 346. This is the first written representation of a decimal positional manner of writing as we do. From Brahmi evolved first the Bakhshali number system and then the Gwalior system, ca. 850 A.D.

6.7 The Gwalior numerals

1	2	3	4	5	6	7	8	9

First the zero was still unknown and not necessary, the result of a calculation first was written as:

.. . .
456 = 456 45 = 450 46 = 406

The earliest appearance of the symbol that we associate with the zero in India in a decimal place-value system is in an inscription from Gwalior dated 'Samvat 933' (876 A.D.) where the numbers 50 and 270 are given by and respectively.

6.8 Multiplication by the erasing method on a abacus [2)]

An Indian technique to do arithmetical calculations was by writing number symbols in sand or dust, on a board, a form of an abacus. On this they wrote with a sharpened stylo or with their fingers Brahmi- or Gwalior numerals and rubbing out intermediate steps as one proceeded. Example of the multiplication 325 x 28 from the period before the sixth century B.C.

	3	2	5
2	8		

6	*3*	2	5
2	8		

6	*3*	2	5
2	*8*		

First multiply the *italic* represented digits, first *3 x 2* and *3 x 8*, the product **bold** represented.

2

6	**4**	2	5
2	8		

8	**4**	*2*	5
➡	*2*	8	

4

8	**4**	*2*	5
	2	*8*	

Cancel *3*, set **2** and **4** and add **6** + **2** = **8** and move 28 to the right, than *2 x 2* and *2 x 8*.

1 6

8	**8**	2	5
	2	8	

8	**9**	**6**	*5*
	➡	*2*	8

1

8	**9**	**6**	*5*
		2	*8*

Add **4** + **4**, cancel *2*, set **6**, add **8** + **1** and move 28 to the right, than *5 x 2* and *5 x 8*.

4

9		**6**	5
		2	8

9		**6**	
		2	8

9	**1**		
		2	8

Add 1 + 9, cancel 5, add 6 + 4.

First the result was still written as:
empty, empty, 1 hundreds, 9 thousands,
later first as 91••, finally as 9100.

6.9 The square and cube root [3)]

Ãryabhata could extract the square and cube root; example 1: extract the square root of 15129. Counting from right tot left, the odd places are called *varga* (o) and the even places are called *avarga* (—). The nearest square root to the number in the last odd place on the left = 1, is set down in a place apart (the 'quotient').

```
o —— o —— o
1 5 1 2 9
-1
0 5
- 4
  1 1
-   4
    7 2
-   7 2
      0 9
-       9
        0
```

the square of 1 = 1^2 = 1
twice the root (2 x 1) = ***2***: 0
2 times ***2*** = 4

1 is the first digit of the root
1 – 1 = 0; now ⇓5; 5 ÷ ***2*** = 2,
2 is second digit of the root.

square of second digit 2^2 =
twice the root (2 x 12) = ***24***
3 times ***24*** = 72

the first 2 digits of the root now are 1,2
72 ÷ ***24*** = 3,
3 is third digit of the root.

square of third digit 3^2 =

if the last subtraction leaves no remainder the square is exact: 123.

Example 2: extracting the cube root: $\sqrt[3]{1860867}$

Counting from right to left the 1st , 4th , etc. places are named *ghana* (cubic) indicated by (o) and the 2nd , 5th , etc. places are called the first *aghana* (non-cubic) places; the 3rd , 6th , etc. places are called the second *aghana* (non-cubic) places indicated by (——). The nearest cube root to the number in (or up to and including) the last *ghana* place on the left = 1, is the first figure of the cube root and is set down in a place apart (the'quotient').

	o —— o —— o 1 8 6 0 8 6 7	the first digit = 1 (= *purva*)
cube of root $1^3 = 1$	- 1	1 is the first digit of the root
3 times square of root $1(3 \times 1^2) = \boldsymbol{3}$	0 8	$8 \div \boldsymbol{3} = 2$ (2 = quotient), the second
2 times $\boldsymbol{3} = 6$	- 6	digit of the root: 12.
square of quotient 2^2 multiplied	2 6	
by 3 times *purva* 1 = 4 x 3 =	- 1 2	*purva* is known part of cube root
	1 4 0	
cube of quotient = $2^3 = 8$	- 8	
3 times square of root $3 \times 12^2 = \boldsymbol{432}$	1 3 2 8	$1328 \div \boldsymbol{432} = 3$ (new quotient)
3 times 432 = 1296	- 1 2 9 6	or next digit of cube root: 123
	3 2 6	
square of quotient 3^2 multiplied	- 3 2 4	
by 3 times *purva* 12 = 9 x 36 =	2 7	*purva* is known part of cube root
cube of quotient = 3^3 =	- 2 7	if the last subtraction leaves no
	0	remainder the cube is exact: 123

6.10 Multiplication by the grating method

In his book *Lilavati* (named after his daughter) Bhaskara (1150) gave five methods for multiplication. The most important was later named the grating method or *Gelosia*, Italian for venetian blinds, after the likeness of the lattice for the windows in Venice. Operation procedure: Figure 6.4 shows how the grating method is used to multiply 5612 by 468. Enter 5612 along the top and 468 along the right side. The product to be entered in each cell is obtained by multiplying the numbers of the row and column in which it lies and arranging the result with the 'units' digit below and the 'tens' digit above the cell's diagonal. For example, in the top-left cell of the square is entered 20, the product of 4 and 5, arranged so that 2 is above the diagonal and 0 below it. When all the entries have been made, the numbers along the diagonal are added starting at the lower right cell 6. Continuing to 2 + 1 + 8 = 11, carrying over if necessary, (1 is written at the bottom and 1 is carried over the third diagonal column), etc. The result can be read as 2,626,416

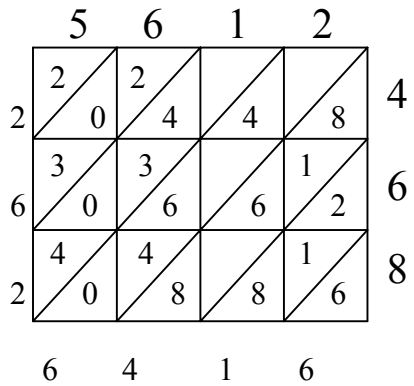


Figure 6.4

1) George Gheverghese Joseph
The Crest of the Peacock; Ancient Indian Mathematics

2) George Ifrah Les chiffres ou L'histoire d'une grande invention

3) Henrietta O. Midonick; collected in: The Treasure of Mathematics:
Ãryabhata, the Elder

Chapter seven, *The Arabs*

The Arabs lived on their peninsula for the largest part in nomadic tribes, known as Bedouins. These tribes did not know private land property and often fought each other for springs and pasturelands for their flocks. This struggle prevented the originate of every larger connection. The family, the relations and the tribe were the only ones who could be trusted. Only in the oasis area Arabs lived in cities, among others in Yemen, Medina and their capital Mecca. The most of the Arabs could neither read nor write. They had a faith with a multitude of gods, polytheism.

7.1 The Prophet Mohammed (580 – 632)

One of them, Mohammed, born in Mecca, came in contact with Jews and Christians, who professed a monotheistic faith. The amalgam of religious feelings that were raised in Mohammed's mind led him to regard himself as the apostle of God sent to lead his people. He created a new monotheistic faith, with as sacred book the Koran, which he claimed had been directly revealed to him by the Archangel Gabriel. For a period of ten years he was preaching in Mecca. In 622 he had incurred the wrath of pilgrims who had come to worship at a shrine called the Caabah – a shrine then dedicated to many gods. Mohammed had to flee to Medina.

7.2 The Hidsj ra

This 'flight', the *Hidsjra*, (Hidsjrat al-nabi = flight of the prophet, 20-09-622) marked the beginning of the Mohammedan era. (The Muslim year is a lunar year and therefore about eleven days shorter than the Western solar calendar year, so an A.H. date (after Hidsjra) cannot be converted to an A.D. date simply by adding 622). The Islam demands the Muslims to protect, not the family or the relations, but the community of all believers against the remaining part of mankind.
Within ten years Muhammad, as a religious and military leader, had established a Muslim state with the centre at Mecca, within which Jews and Christians, being also monotheistic, were afforded protection and freedom of worship.

7.3 The caliphs

Mohammed had not arranged his succession. After his sudden death in 632, some 'false prophets' had presented themselves. The first caliph (successor of Mohammed) Omar Aboe Bekr had to re-unite Arab with force. Some of the military attacks in border areas were so successful that they overran neighbouring territories with astonishing rapidity. Within a few years Damascus and Jerusalem and much of the Mesopotamian Valley fell to the conquerors and by 641Alexandria, which for many centuries had been the mathematical centre of the world, was captured. When the Arabs took Alexandria, the worth of this was still not recognised. At its height, the establishment of Islamic rule over areas stretching from North Africa in the south to the borders of France in the west, right across Persia and the Central Asian plains to the borders of China in the east, and down to Sind in northern India. Much of this vast territory was brought under Islamic rule in less than a hundred years. As the Arabs conquered a country, they would often take over the mode of writing, particularly the notation of numerals, of the conquered people and search out the sources of knowledge in the literature that may have survived the ravage of war. In days of war the Muslims were merciless, in time of peace cultivated and tolerant. The culture of the conquered people was not destroyed but absorbed in the flourishing culture of the Islam.

7.4 The schism in East and West.

The caliphs belonged to one of two different dynasties.

The Umayyad, the early rulers of the Eastern empire, (660 – 750) came from the Arabian Peninsula, with their capital at Damascus, but in the year 750 they were overthrown and power passed to the Abbasids.
The Abbasids were more cosmopolitan, welcoming new converts from many different ethnic groups. See the House of Wisdom.

7.5 Moorish Spain

Among the few of the Umayyad, who could escape, was a young man of twenty named Abel Rahman, who reached Spain and re-established Umayyad power there. The whole Iberian peninsula was already conquered on the Visigoths. (In 732 Charles Martel between Tours and Poitiers conquered the Muslims in France). For the next three centuries Spain was the centre of Muslim power in the West, with its political and intellectual capital at Córdoba. In 1085 Alphonso VI of Leon recaptured Toledo from the Moors.
As from the separation of 750, Moorish Spain was far backward to the East-Arabs: the first university of Córdoba was founded in 976, two centuries after that of Baghdad (It was there where Gerbert d'Aurillac in ca. 997 met Hindu numerals); the first paper factory in Spain 350 years after that of the East-Arabs.
The number system of the Arabs did not answer to rule a large empire and they borrowed the Ionic-system ($\alpha\beta\gamma$) from the Greek. In 706 caliph Walid prohibited the Greek language in favour of the Arabian, but had to allow to make calculations with the Ionic-system.

7.6 The House of Wisdom

In 762 the second of the Abbasid caliphs, Aboe Jafar al-Mansur (754 – 775), moved his capital to Baghdad and began the process of transforming it into a new Alexandria. This ambitious programme of construction was carried out during the caliphates of Haroun al-Raschid (786 – 809), and his son al-Mamun (809 – 833), and included an observatory, a library, and an institute for translation and research named *Bait al-Hikma* (House of Wisdom) that was to be the intellectual centre of the Arab world for the next two hundred years. It housed translators, busy rendering into Arabic scientific classics written in Sanskrit, Pahlavi (the classical language of Persia) Syrian and Greek, among them as well Jews, Nestorian Christians as pagans. Early collections included Greek manuscripts from the Byzantine Empire, translations of Babylonian astronomy by the Syrian schools of Antioch and Damascus, and the remains of the Alexandrian library in the hands of the Nestorian Christians at Edessa.
In 773 the Indian Kanaka, well versed in astronomy, came to Baghdad as a member of a diplomatic mission from Sind, in northern India. He brought with him Indian astronomical texts, including *Surya Siddhânta* and the works of Brahmagupta.
During the caliphate of Haroun al-Araschid (known through the *Arabian Nights*) a part of *Elements* of Euclid was translated.
During the caliphate of al-Mamun Arabic versions have been made of all the Greek works he could lay his hands on, including Ptolemy's *Megalè Syntaxis* (Great System), known under the Arabian name *Almagest,* and a complete version of Euclid's *Elements*, Babylonian algebra, astronomy and astrology and Indian works. Al was studied, multiplied and distributed.
For six centuries the Arabs continued to be the most serious students in the world. They kept the torch of learning aglow while Europe was in darkness.

The House of Wisdom (like its predecessor in Alexandria) was completely destroyed in 1258 by the grandson of Genghis Khan, Hulagu Khan, who destroyed Baghdad together with all its libraries and museums.

7.7 Abu J afar Mohammed ibn-Moesa al-Chwarizmi (ca. 780 – 850)

One of the scholars, who worked at the House of Wisdom, was Abu Jafar Mohammed ibn-Musa al-Khwarizmi (to give him his full name, which means Mohammed, the father of Jafar and the son of Musa, from Khwarizm), the modern city of Khiva in Uzbekistan, south of the Aral Sea. Besides treatises on the astrolabe and sundial, he wrote a number of books, on astronomy and arithmetic. The two most influential are: *Hisab al-jabr w'al- muqābalah* and *Arithmetic,* whose original Arabic version no longer exist – we know it only in Latin translation (by John of Seville and Robert of Chester): *Algorithmi de Numero Indorum:* Concerning the Hindu Art of Reckoning, based on an Arabic translation of Brahmaupta. In *Arithmetic,* he distinctly indicated that the system came to him from the Indians. In *Arithmetic* is found the first systematic treatment of arithmetical operations; Al-Khwarizmi discusses the digits, the use of the zero, the place-value system and rules for performing the four arithmetical operations. The use of the Hindu numerals quickly spread through out the entire Arab Empire. Al-Khwarizmi gave so full account of the Hindu numerals that he probably is responsible for the widespread but false impression that our system of numeration is Arabic in origin. The new notation came to be known as that of al-Khwarizmi, or more carelessly algorismi; ultimately any peculiar rule of procedure or operation came to be called simply algorism or algorithm.
The word *al-jabr* from the title of his book *Hisab al-jabr w'al- muqābalah* presumably meant something like ‘restoration’ or ‘completion’ and seems to refer to the transposition of subtracted terms to the other side of an equation.
Al- muqābalah is said to refer to ‘reduction’ or ‘balancing’ that is the cancellation of like terms on opposite sides of the equation.
This *al-jabr* has come down to us in a version from a Latin translation: *Liber algebrae et almucabala.* For it is from the title of his book that Europe later learned this branch of mathematics, the word algebra bearing this name.

7.8 Multiplication with Ghobar-numerals Arabic erasing method

Two Arabic systems of numerals were derived from the original Brahmi numerals, an East Arabic form used by Arabs in the Middle East which is very similar to their present forms, and a West Arabic form used by Arabs in Spain from which our numerals developed.
The Western version of these numerals were called Gobar numerals – presumably because, as the meaning of Gobar suggests, these symbols were written on a board containing sand or dust, a practice that was popular in India. Doubling and halving was often applied, an inheritance from Egypt. In the following example the multiplicand 5612 is to be multiplied by 72, the multiplier. The most left figure of the multiplier is to be written under the most left figure of the multiplicand. The partial products are written in **bold** above the multiplicand:

					10
		35			**35**
*5*612	first 7 x *5* = **35**	*5*612	next 2 x *5* = **10**		*5*612
72		72			72

erase *5* of *5*612, add the partial products and move 72 over one place to the right;

				12
		4 2		**4 2**
now	7 x *6* = **42**	**36** *6*12	and 2 x *6* = **12**	**36** *6*12
		72		72

erase *6*, again add the partial products and move 72 over one place to the right;

				2
		7		**7**
next	7 x *1* = **7**	**4032** *1*2	and 2 x *1* = **2**	**4032** *1*2
		72		72

erase 1, once again add the partial products and move 72 over one place to the right;

				4
		1 4		**14**
finally	7 x *2* = **14**	**40392** *2*	and 2 x *2* = **4**	**40392**
		72		result = **404064**

7.9 Division according to the Hindu erasing method [1)]

This Hindu method of division proves to be very handy when work is done on a dust table.
Consider as an example 46468 ÷ 324
The 'guess quotient' of 464 ÷ 324 is 1; set 324 under 464 and 1 above the 4:

1			
46468	1 x 3 = 3	4 – 3 = **1**	**1**6468
324	1 x 2 = 2	6 – 2 = **4**	1**4**468
	1 x 4 = 4	4 – 4 = **0**	14**0**68

1406 ÷ 324 = 4; set 4 next to the 1 of the quotient and move the divisor one place to the right:

14			
14068	4 x 3 = 12	14 – 12 = 2	**2**068
324	4 x 2 = 8	20 – 8 = 12	**12**68
	4 x 4 = 16	26 – 16 = 10	1**10**8

1108 ÷ 324 = 3; set 3 in the quotient and move the divisor one place to the right:

143			
1108	3 x 3 = 9	11 – 9 = 2	**2**08
324	3 x 2 = 6	20 – 6 = 14	**14**8
	3 x 4 = 12	48 – 12 = 36	136

Result: 46468 ÷ 324 = 143, remainder = 136

7.10 Division with the crossing out method / galley method [2)]

Abul Hassan al-Uqlidisi made a living by copying the works of Euclid (hence the 'Uqlidisi'!). In his book *Kitab al-fusul fil-hisab al-Hindi*: *The Book of Chapters on Indian Arithmetic*, written in Damascus in the year 952, operations are given in both the Indian decimal place-value system and the Babylonian sexagesimal system. His book is particularly notable for the first use of decimal fractions in computing with the new numerals. He adapted the Indian dustboard techniques of computation to methods suitable for pen and paper. It was a common practice, both in India and in the Arab world, to do arithmetical calculations by writing numbers in sand or dust, rubbing out intermediate steps as one proceeded. Al-Uqlidisi had strong reservations about this procedure, not for any shortcomings in the method itself, but

because dustboard calculations were carried out by street astrologers and other good-fornothings to earn their livelihood! This rubbing method is probably of Hindu origin. Because of the shape of the final result of the division of this method, it was later called the *galley* method (after a boat or galley with its sail spread).

```
37 | 5612 | 1
     37
```

Example: 5612 ÷ 37 Step one: 56 ÷ 37 = **1**, write this first figure of the quotient to the right and 37 under 56;

```
     2
37 | 5612 | 1
     37
```

subtract: 5 – 3 = 2 , write the difference 2 above 5 and crossing out the numbers used: 5 and 3;

```
                1
                29
          37 | 5612 | 1
                37
```

subtract: 26 – 7 = 19 , write 19 above 26 and crossing out 2 , 6 and 7.

```
     1
     29
37 | 5612 | 15
     375
     18
```

Step two: 191 ÷ 37 = **5**, write this second figure of the quotient next to the 1 and the product 185 under 191.

```
     1
     29
37 | 5612 | 15
     375
     18
```

subtract: 1 – 1, and crossing out these two ones;

```
                1
                29
          37 | 5612 | 15
                375
                18
```

next 9 – 8 = 1, write 1 above 9 and crossing out 9 and 8;

```
                1
                29
          37 | 5612 | 15
                375
                18
```

and also 11 – 5 = 6, write 6 above 1 and crossing out 11 and 5.

```
     11
     296
37 | 5612 |151
     37157
     183
```

Step three: 62 ÷ 37 = **1**, write this third figure of the quotient next to 15 and the product 37 under 62;

```
                113
                296
          37 | 5612 |151
                37157
     2          183
     113
     2965
37 | 5612 | 151
     37157
     183
```

subtract: 6 – 3 = 3, write 3 above 6 and crossing out 6 and 3;

finally: 32 – 7 = 25, write 25 above 32 and crossing out 32 and 7.

```
                2
                113
                2965
          37 | 5612 | 151
                37157
                183
```

The quotient is 151 and the remainder is 25.

7.11 The theory of numbers [3)]

In *Kitab al-shifa* (Book of Physics) van Ibn Sina (or Avicenna, as he came be known in Europe), contains sections on arithmetic, among others an explanation of the rule for casting out nines, and some rules for summing a square and triangle arrays of odd numbers.

The rule for casting out nines: an originally Indian method of checking addition and multiplication. It uses the property that the sum of the digits of any natural number when divided by 9 produces the same remainder as when the number itself is divided by 9. For example, in checking that the product of 436 and 659 is 287,324:

			casting out nines
436	4 + 3 + 6 = 13	1 + 3 = 4	leaves remainders of
x 659	6 + 5 + 9 = 20	2 + 0 = 2 x	4, 2 and 8 and since
287324	2 + 8 + 7 + 3 + 2 + 4 = 26	2 + 6 = 8	4 x 2 = 8, the multi-
			plication is correct.

Rule 1: If successive odd numbers are placed in a square table (see examples), the sum of the numbers lying on the diagonal will be equal to the cube of the side: 1+9+17 = 27 or 5+9+13 = 27 = 3^3 and 1+13+25+37+49 = 125 or 9+17+25+33+41 = 125 = 5^3 and the sum of the numbers filling the square will be the fourth power of the side: 1+3+5+7+9+11+13+15+17 = 81 = 3^4 and 625 = 5^4

5 3 1		9 7 5 3 1
11 9 7	or	19 17 15 13 11
17 15 13		29 27 25 23 21
		39 37 35 33 31
		49 47 45 43 41

There is a clear implication here that Ibn Sina knew that the sum of successive odd numbers starting with 1 is equal to the square of the number of odd numbers being added. For example: 1+3+5+7+9+11=36 = 6^2.

Rule 2: is for summing a triangular array of odd numbers:

1
5 3
11 9 7
19 17 15 13
29 27 25 23 21

If successive odd numbers are placed in a triangle, the sum of the numbers taken from one row equals the cube of the row number: 5 + 3 = 8 = 2^3 and 29 + 27 + 25 + 23 + 21 = 125 = 5^3 etc.

1) Ewald Fettweis
Wie Man einstens rechnete

2) Arthur Gittleman
History of Mathematics

3) George Ghever Ghese Joseph
The Crest of the Peacock
Non-European Roots of Mathematics

Chapter eight, *Latin Europe from 1000 to 1600 A.D.*

8.1 Counting boards of the Late Middle Ages

Because of the restoration and great expansion of trade about the 12th century (as a result of the Crusades and the Hanseatic League, among other things), there was an enormously increased demand for money changing. Since the time of Gerbert and his disciples the old counting board with vertical columns (and apices) had been rotated through a quarter of a turn into one with horizontal lines. This happened perhaps in the 13th century at about the time when metal counters came into use.
Merchants and money-changers again took up the old, undifferentiated *calculi*. Arithmetic in counters does not only serve for them who cannot write and read. Merchants also used counters for calculations, which not could be performed mentally. These stamped or embossed counters, however, had no monetary value. The counting boards could have lines of unspecified value, or of specified value, that means, for the use for calculations about money or measures.

8.2 The counting board for unspecified value

The horizontal counting board with lines of unspecified value was intended for true computations with abstract, unnamed numbers, and could also be used for multiplication and division.
Four parallel horizontal lines were drawn on a board or table, with a vertical line down the middle dividing them into two columns or *bankire*, see Figure 8.1, showing the basic form of such a board. The topmost of the four horizontal lines was marked with an X at its centre. The counters were placed upon the lines and in the spaces between the lines. A counter at the bottom line had the value of 1, on the second line 10, on the third line 100, and on the top line, which was marked with the X, 1000.

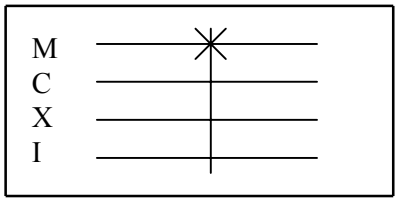


Figure 8.1

For larger numbers lines were added and nominated as ten-thousands, hundred-thousands, and thousand-thousands, with a next cross.
A counter between two lines has the value of 5 times the counter on the line below or half the value of the counter on the line above. The value of the counters has a similarity with those of the Salamis counting board, see Figure 3.3 page 19. The Roman hand abacus of Figure 4.1 page 26 also has ‘beads’ which have 5 times the value of the ‘beads’ or calculi below.

A counting board was not always required, one could draw lines on a table with chalk or charcoal and on the failure of counters one could use change or even pebbles. According to a source from 1432, the Chamber of Accounts in Holland purchased:

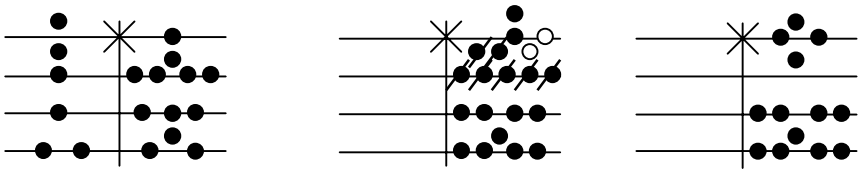
vier ellen groens lakins dair die meijsters vander rekenijnge over rekenden, doe zij die rekenijnge vanden dienres van Hollant hoorden.

Four ell (yard) of green woollen cloth on which the masters of the accounts computed, when they verified the bills of the servant of Holland.

A reckoning cloth could easily be carried about by the officials whose duty it was to check the calculations made by the mayors of the towns throughout the country.

Many textbooks of the time list the following branches, or operations, of arithmetic: *numeration* (the placing of the numbers on the counting board), *addition*, *subtraction*, *duplication* and *mediation* (that is, doubling and halving of numbers), *multiplication*, and *division*. Furthermore, along with each of these operations there was *elevation* (in French déjeter), or 'purification', the grouping of units into smaller numbers of higher units; and with subtraction there was also *resolution*, the 'de-grouping' of higher units into lower ones. Both procedures together constituted the operation of *reduction*.

8.3 Addition on a counting board with unspecified lines example: ad 5612 to 1937



Start elevation at the bottom with the smallest value.

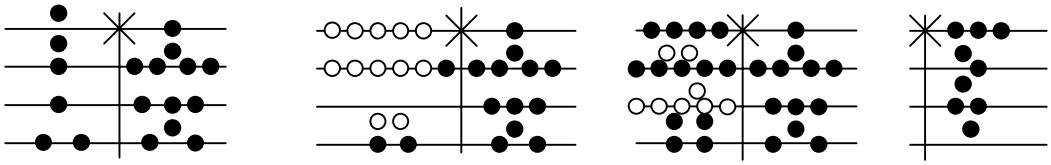
5612 + 1937 — Elevatio — sum = 7549

numeratio — elevation, or purification: the counters with a stroke through ⊘ place the counters — them were elevated to a ○ in the space or on the line above:

Schau wo zwen pfennig im spatio	See where two counters lie in the space
Ligen hebs auff und leg ayn do	Pick them up and lay one down
Auff d'negsten lyni ubersich /	On the next line above /
Desgleich so fünf liegen halt dich	Likewise if five counters lie
Auff eyner lyni merk und guck /	On one line, take note and look /
Ins spatium drüber ayn ruck.	And put one in the space above.

Numbers and results were noted in Roman numerals!

8.4 Subtraction on a counting board with unspecified lines example: 5612 — 1937



5612 — 1937 — Resolutio — 2e time — subtractio

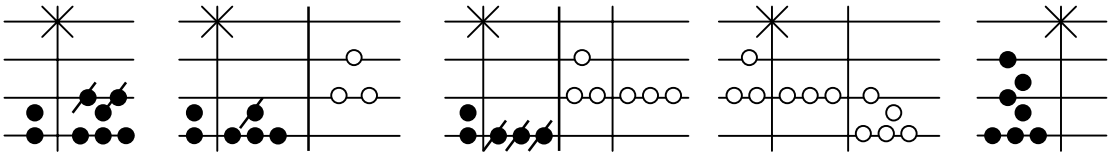
Numeratio — de-grouping of the largest number, — drawing away

place the counters — start with the highest value (above). — result = 3675

Or, in this problem, after picking up all possible counters, (of the same value), only the partial problem 5000 – 1325 remains to be solved.

8.5 Multiplication on a counting board with unspecified lines:

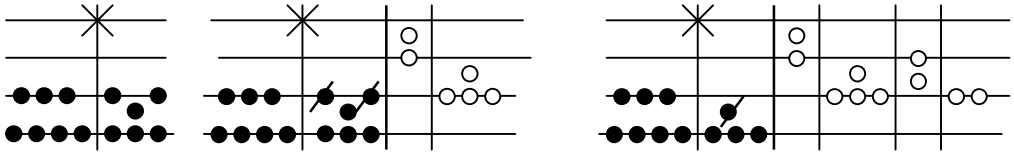
a) with a multiplier < 10, example: multiply 6 by 28.



6 x 2(0) = ... 12(0) 6 x 5 = 3 x 10 = 30 6 x 3 = 18

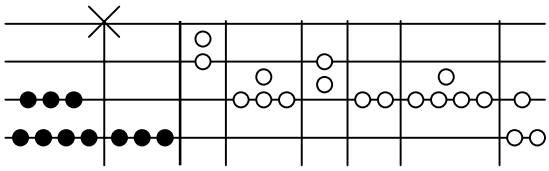
finally all the ○ have to join and elevate, product = 168

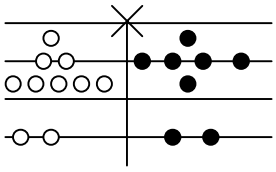
b) with a multiplier and multiplicand both with two figures, example: multiply 34 by 28



3(0) x 2(0) = ...6(00) 3(0) x 5 = ... 15(0)

4 x 2(0) = ...8(0) 4 x 5 = ... 20





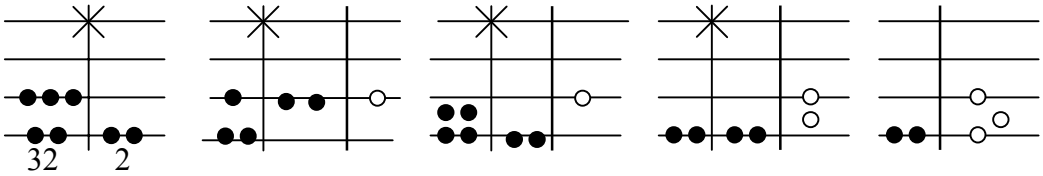
3(0) x 3 = ... 9(0) product = 952

4 x 3 = ... 12

Finally, again add all ○ , (this can be done as soon as possible).

8.6 Division on a counting board with unspecified lines

e.g. 32 ÷ 2

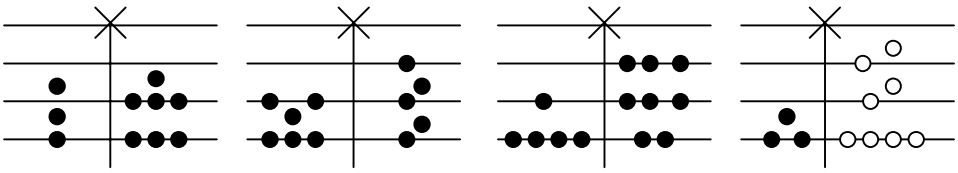


Take away as often as you can the same number of counters (2) from the same line, start on top (3) and place each time a counter ○ to the proper line. resolvatio again quotient

Instead of multiplication often doubling (duplicatio) and halving (mediatio) was used.

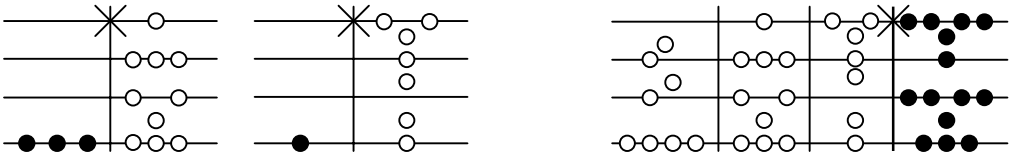
8.7 Doubling and Halving

These operations were first introduced as separate, independent procedures. It was not until the 16th century that they were generally recognised as special cases of multiplication and division, respectively. Example: multiply 56 by 83 using doubling and halving.



56 x 83 28 x 166 14 x 332 7 x <u>664</u>

56 ÷ 2 = 28 etc. until in the left column appears an odd number, here 7. Than 7 ÷ 2 = 3 ½. Now, this ½ is ignored, thus 7 ÷ 2 = 3. After the halving has been carried all the way down to 1, all the doubled number with odd multiplies (underlined) are added together.



3 x <u>1328</u> 1 x <u>2656</u> 664 + 1328 + 2656 = 4648

3 ÷ 2 = 1 ½, again ignore this ½

14 x 332 = 7 x 664 = 3 x (2 x 664) + *1 x 664* = 1 x (2 x 1328) + *1 x 1328* + *1 x 664.*

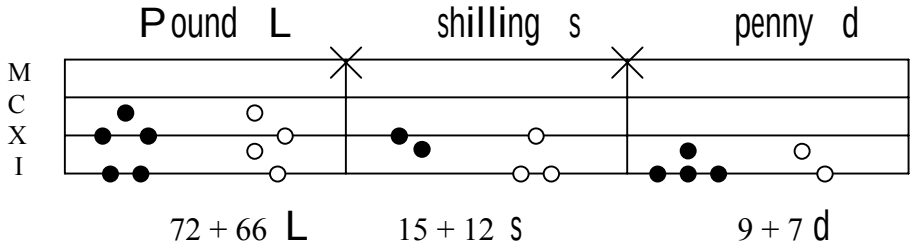
The procedure of doubling, of course was just the reverse.

8.8 Computing on a counting board with specified lines for coins

The counting board with specified ‘rows’, each for a specific denomination, was used only for calculations involving sums of money, primarily the addition and subtraction of receipts and outlays and the conversion of one denomination into others (such as Heller or farthings into pennies, shillings, pounds, or gulden). Such a counting board is shown in Figure 8.2.

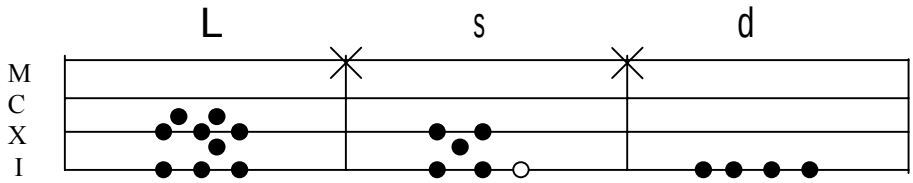
Figure 8.2 A line board with columns for various coins, probably from Strasbourg

The counting bank was divided into designated columns, each coin denomination was placed in its column and added together. Then 12 pence were converted into shillings and 20 shillings grouped into pounds. Or 12 pennies were converted into groschen and 20 groschen grouped into gulden.

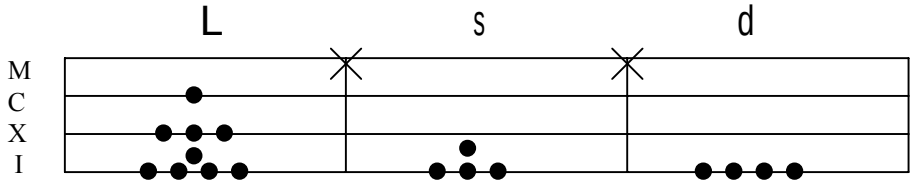


72 + 66 L 15 + 12 s 9 + 7 d

9 pence is (6 + 3) and 7 is (6 + 1)!



Start with the pence: 12 pence give 1 shilling and 20 shillings give 1 pound.



Result: 139 pounds, 8 shillings and 4 pence.

8.9 The Exchequer tables of the British Royal Treasury The Auditors Use

The word *exchequer*, and the French *échiquier*, was derived from the Latin word *scaccarium*, the 'checker'– board, which was in turn a Latin translation of the Persian word šah, 'king', the chief piece of the game of chess, which is played on a 'chequered' board. The origin of the English word Exchequer, referring to the British Royal Treasury, goes back to the time of Henry I. In the year 1110 Henry I addressed a writ to the barons of the exchequer, *baronibus de scaccario,* for the collection of a tax of three shillings per hide for the marriage of his daughter. Apparently the scaccario or exchequer was already in being.

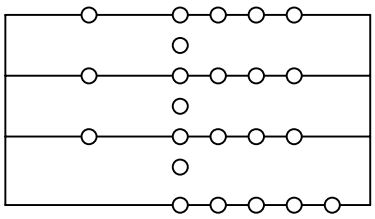
Henry II (1133 – 1189) in whose reign half of France was subject to the English Kings, ruled from 1154 to 1189. Fortunately we are adequately informed about the procedures of the court of audit by the *Dialogus de Scaccario* – Dialogue Concerning the Chessboard, written in 1186 by the then Royal Treasurer Richard, Bishop of London.
The officials who were charged with the administration of the state's receipts and expenditures used to gather around a table which was covered with a woollen cloth marked by perpendicular lines into a chequered pattern. In early times taxes were paid, not in money but in goods and the counters had various forms and colours which represented the different goods. The *Dialogus de Scaccario* of 1186 says:

…*Superponitur scaccario pannus non quilibet, sed niger virgis distinctus… In spatiis autem calculi fiunt inxta ordines*… On the reckoning table was placed not just any cloth, but a black cloth marked with parallel lines at equal distances…In the spaces between the lines the counters were placed according to the rules.

The order of the columns of the reckoning table from the *Dialogus de Scaccario* from right to left was pence, shillings, £1, £20 (a score), £100, and £1000, just as notches were carved in the tallies for these very same amounts. See Figure 15.3.
The *chequer* counting board was used in conjunction with the tally sticks on which the amounts of the Exchequer were recorded. Thus the act of comparing the board with the tally sticks was called to *check*. Operating on this chequer board was described by Robert Recorde as the *Auditors Use.*

8.10 The Merchant's Use

Merchants seem to have used a rather simpler form of the same system for their smaller everyday amounts. The *Merchant's Use* shown by Recorde had the values set out in this manner:



Scores of Pounds	£100	£20	£20	£20	£20
		£5			
Pounds	£10	£1	£1	£1	£1
		5s			
Shillings	10s	1s	1s	1s	1s
		6d			
Pence		1d	1d	1d	1d

Robert showed no lines but no doubt they were necessary.

In Britain the minister of finance of Her Majesty is still called the Chancellor of the Exchequer: the Chancellor of the Chessboard.

An English chequered counting board from the 14th century is shown in Figure 8.3.

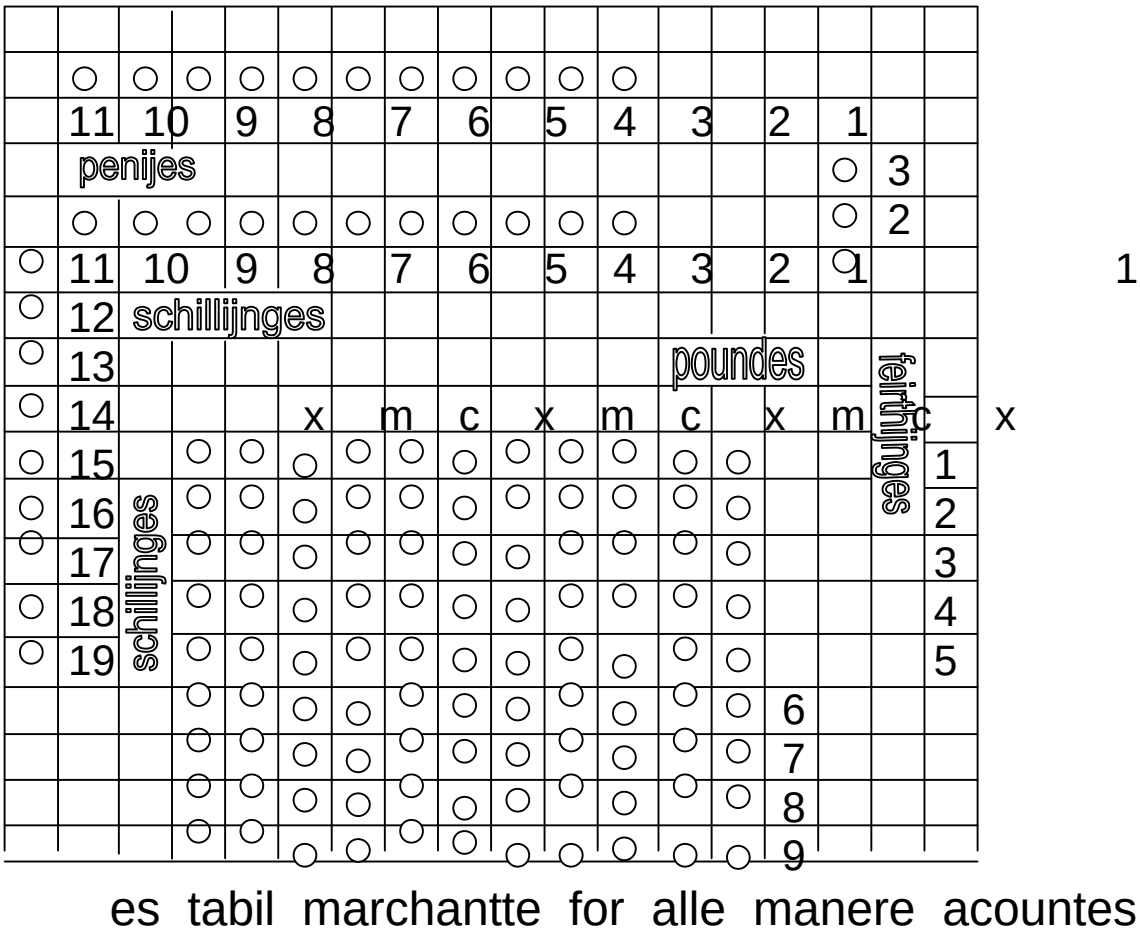


Figure 8.3 According a drawing from a manuscript from the 14th century.

On this English chequered counting board, spaces were provided for the four English denominations: 1pound = 20 shillings, 1 shilling = 12 pence, 1 penny = 4 farthing.

8.11 The number tree

An interesting variation was sometimes used, particularly in France, in which there were no actual lines on the table, the places being marked by a vertical row of *jetons* or counters, the *Arbre de Numeration* or number tree. Jetons replaced the lines on the board, which were set down and not moved again. This custom was followed by Le Gendre in France. In 1753 he still found it necessary to add a chapter to his *L'Arithmetique en sa Perfection* describing this way of calculation par les jetons.

8.12 Hindu-Arabic numerals at the end in Latin Europe

One cannot absorb the wisdom of one's neighbours if one cannot understand their language. The Moslems had broken down the language barrier to Greek culture in the 9th century. The revival of learning in Latin Europe took place during the crusades from 1096 to 1272.
The discovery by Crusaders that the Moslems possessed a great store of knowledge set Europe buzzing, and to tap this new source of information scholars set out for those places at which contact between the Christian and Islamic civilisation was most intimate.
The most obvious point of contact from which the Arabic materials were passed to the Latin West was the Spanish peninsula. Spain's doors were opened when, in 1085, the Christian Alphonso VI of Leon recaptured Toledo from the Moors. Now the language barrier to Arab

science had broken down. At Toledo there arose a regular school of translation of Arabic books of science. The work of translation was extremely awkward. First the Arabic text had to be read aloud, then rendered into Hebrew by a Spanish Jew or into current Spanish; and finally a Christian translator turned this into Latin. Moreover, Medieval Latin was not yet equipped with an adequate supply of technical terms, so that the meaning of some of these in Arabic was imperfectly known to the translators themselves.
Here follows the best known translators.

8.12 1 Adelard of Bath (ca 1075 – ca 1160) was an English monk who travelled far and wide – to Spain, Southern Italy, Sicily, Greece, Syria, and Palestine – seeking out the knowledge he had heard of. About the year 1120, he had produced a Latin text whose first words *Digit Algorismi* (Thus said Algorismi) resulted in this new science being known as algorism. (This term spread through every European language to the point where the process of doing arithmetic with the Hindu-Arabic numerals was termed *algorism*, and this in turn has given us the term *algorithm* which is so familiar to users of computers). In 1126, he also had translated al-Khwarizmi's astronomical tables. In 1142 he translated Euclid's *Elements* from the Arabic into Latin. In this way he made the geometry of the great Alexandrian known for the first time in Latin West. Later, in 1155, he also translated Ptolemy's *Megalè Syntaxis* (*Almagest*) from Greek into Latin.

8.12.2 Gerard van Cremona (1114 – 1187) was especially interested in astronomy. Ptolemy's works were not available to him, so Gerard was drawn to Toledo, where he learned Arabic from a native Christian teacher. Gerard devoted the rest of his life to translate scientific works from Arabic into Latin. He is credited with having produced Latin versions of no fewer than 90 complete Arabic texts. Among these was a revised version of Thabit ibn-Qurra's of Euclid's *Elements* and *Algebre* of al-Khwarizmi. In 1175 he produced the first Latin version of Ptolemy's *Almagest*. It was chiefly through this work that Ptolemy became to be known in the West.

8.12.3 Alexander de Villa Dei (? - 1240) was a native of Normandy. He was already very well known for a Latin Grammar, also in verse, which was widely used in schools of the period. In about 1220 he produced *Carmen de Algorismo*, (The Poem of Algorism), presenting the ten Hindu-Arabic numerals, and the basic operations of arithmetic in hexameter verse. Being only 284 lines long, it was easily copied by scribes. Therefore it became very popular, being copied many times, and used frequently at universities.

8.12.4 J ohn of Halifax(ca 1200 – 1256) was also known as John of Holywood or **Sacrobosco**, the Latin translation of Halifax. He was educated at Oxford, and one of the most celebrated teachers at the University of Paris. He composed an introduction to the ten new Hindu-Arabic numerals and their use in the basic operations of arithmetic.
In his *Algorismus vulgaris*, (Common Algorism), only about 4,000 words long, and therefore very popular, he presented these numerals as follows, see Figure 8.2.

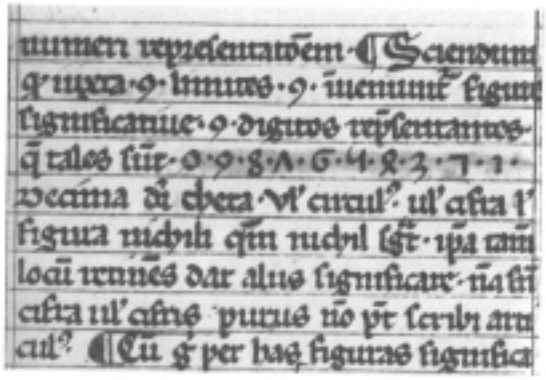


Figure 8.2

Hundreds of copies of both *Carmen de Algorismo* and *Algorismus vulgaris* were made by students, undoubtedly taken by dictation; the master would read two or three lines, the students copying, and then the master would discourse up on the meaning.

However, no single work had such influence as:

8.13 Liber abaci by Leonardo of Pisa

The first great attempt to introduce the Hindu-Arabic numerals was made by Leonardo of Pisa (1170–1240), also known as **Fibonacci**, son (figlio) of Bonaccio, the nickname of Leonardo's father, who was the governor of the Pisan trading station at Bugia in Algeria. Leonardo was allowed by his father to travel to Bugia in order to take up the *studium abaci* – to learn the art of computation with the nine Hindu numerals. In 1200 Leonardo returned to Pisa. Leonardo published his book *Liber abaci*, written in Latin, in 1202. Although the title of his 'Book of the Abacus' could point to an explanation about computing with counters on a board, his book exclusively introduced the new Hindu–Arabic numerals and an explanation of their use.

Liber abaci was a massive tome for its day consisting of 455 pages divided into 15 chapters. Chapter 1 through 7 dealt with an explanation of counting on the fingers, and introduced the Arabic notation, the place value system, the four fundamental operations on integers and fractions, in which he followed the 'Arabic' style in placing the fraction in front of, not after the integer: ½2 instead of 2½. Chapter 8 to 11 dealt with various applications on commercial arithmetic – *practica*, as it was known in the Middle Ages – the rule of three and the chain rule. The remainder was devoted to methods of calculation involving series, proportions, square and cube roots, and a short discussion of geometry and algebra.

It is one of the ironies of history that the chief advantage of positional notation – its applicability to fractions – almost entirely escaped the users of the Hindu-Arabic numerals for the first thousand years of their existence. In this respect Fibonacci was as much to blame as anyone was, for he used three types of fractions – common, sexagesimal, and unit – but not decimal fractions. In the *Liber abaci*, in fact, the two worst of these systems, unit fractions and common fractions, are extensively used. *Liber abaci* includes tables of conversion from common fractions to unit fraction. Instead of writing $11\frac{5}{6}$, for example, he wrote $\frac{1}{3}\ \frac{1}{2}\ 11$, with juxtaposition of unit fractions and integers implying addition. An example of a division is: Divide 83 by $5\frac{2}{3}$ first 5 x 3, add 2 gives 17; than 83 times 3 gives 249 and finally divide 249 by 17, result $14\frac{11}{17}$. Moreover, problems of the following type abound: If $\frac{1}{4}\ \frac{2}{3}$ of a rotulus is worth $\frac{1}{7}\ \frac{1}{6}\ \frac{2}{5}$ of a bizantium, then $\frac{1}{8}\ \frac{4}{9}\ \frac{7}{10}$ of a bizantium is worth $\frac{3}{4}\ \frac{8}{10}\ \frac{83}{149}\ \frac{11}{12}$ of a rotulus. Pity the poor medieval businessman who had to operate with such a system!

Liber abaci was the most extensive treatment, which also explains why it never became popular. *Liber abaci* was printed in Italy only in 1852!

8.14 Growing need for ciphering

Trade was increasing in the 11th and 12th century and towns were developing around the trading centres. The economy was tending to be based on money, so the feudal landowners were no longer the most powerful men. The power belonged to the wealthy men who could pay their armies with money, rather than land.
An improved rudder was put into use, enabling commercial ships to travel even greater distances. German and Dutch groups were important in trade in the northern European countries, and they formed the Hanseatic League, a group devoted to protecting trading interests in foreign countries. Thus there was also a need for astronomical tables for navigation (to calculate longitude, for example) which, in turn, required trigonometric tables. Many people were interested in astrology and this influenced the demand for astronomical

tables, since the precise locations of the planets at any time were needed to make astrological predictions.
The growth of commerce demanded facility with numbers, if only to prevent one to be cheated by those whose numeracy was mixed with larceny. Robert Recorde had declared the "profit of arithmeticke." It was "needful for all Merchants, Auditors, Treasures, Receivers, and such like, whose Offices without Arithmetick are nothing."

8.15 Education in the Low Countries [1)]

To the end of the 11th century there was only teaching at monastery- and cathedral schools. From ca. 1200 parish-churches in the cities started providing a school. Although these parish-schools did not prepare to clergymen, the church continued to use their great influence. Students, boys and now also girls, learned all by heart. Books were scarcely and ready knowledge by memorising was the only teaching method. Latin church-hymns and prayers and the first principles of Latin grammar were studied. Arithmetic teaching was out of the question.
In the course of the 14th century an increasing number of parish-schools passed in the hands of the city governances. The school became city-school. There, pupils from 8 to 10 years, learned reading and writing in Dutch language. Then the schools were divided into small, low, write, elementary or *Dietse* school and the secondary or Latin schools where the instructions were given in Latin. Instruction in arithmetic at the Latin school still was of an unpretendingly level.
In the course of the 15th century, city-schools became competition of *by-schools*. These were private schools for, or even private education to, children from patricians and wealthy merchants who wanted better arithmetic instruction for their children. By-schools were permitted if no Latin was instructed. In Leyden a schoolmaster requested:

…te mogen gaen leeren leesen, spellen ende scryven als schoolmeester langs de huysen of te mogen schoolhouden der goeder luyden kinderen te leeren leesen, scryven, reeckenen ende chyfferen.
…permission to teach reading, spelling, and writing as a schoolmaster along the houses or to run a school for the wealthy people's children to teach reading, writing, arithmetics and ciphering.

Some of the by-schools became *French-schools*. Besides French, of growing importance in the increasing international trade, also bookkeeping and commercial arithmetic was given at these schools, occasionally German, English, Spanish and/or Italian. A teacher at a French-school wanted the pupils:

…walsch leeren ende scryven ende huer penningen reeckenen ende legge de ghemeyn maniere, ghelyc coipluden ghemeynlic doen.
…teaching Walloon and write and placing counters and reckoning the common way as merchants usually do.

Who had walked through the Latin school could study at a university.
University originally meant *group* as in a *guild*, because the students were united to prevent their exploitation by local landlords and shopkeepers who could charge outrageous prices, due to the large demand for goods and services from the students in the town, unless checked.
The first university in the Low Countries was founded in 1425 at Louvain, the second in 1575 at Leyden.

In British grammar schools arithmetic was rarely taught, but appeared in private schools for a separate fee.

8.16 Arithmetic books in the Low Countries, [2)] Germany, Italy, and England

The oldest Dutch manuscript of arithmetic was written in 1445 in Hildesheim by a certain Bernard, entitled: *Algorithmus is een aert in den welken siin ghevisiteert IX figuren.*
Algorithmus is an art in which are examined IX figures

Every page of the Bamberg block-print book of arithmetic of 1470 was cut entire from a single block of wood. It contained 14 leaves, size:9 x 10 cm.

The oldest Italian printed book of arithmetic was published in Treviso in 1478. The firs page starts: *Incommincia una practica molto bona et utiles a ciaschaduno chi vuole uxare larte dela merchadantia, chiamata vulgarmente larte de labbacho.*
Here begins a very good and useful book of instruction for everyone who wishes to learn the mercantile art, which is popularly known as the art of the abacus.

The oldest fully preserved German printed book of arithmetic was published in Bamberg in 1483, written by the Nuremberg teacher Ulrich Wagner.

The Italian Luka Paciolo published in 1494 his *Summa d'Arithmetica.* (See sector 8.17.5)

The German Cartesian Prior Gregor Reisch published in 1503 *Margarita Philosophia* (Pearl of Philosophy).

The oldest printed book of arithmetic in Dutch is dated 1508, reprinted in 1510, entitled:
Die maniere om te leeren ciifferen ende rekenen metter pennen ende metter penningen. Na de gherechte conste Algorismi. Int gheheele ende int ghebroken.
The way to learn cipher and to reckon with the pen and with the counters. After the true art Agorismi. In whole and in fractions.

Adam Riese (1492 - 1559) who is still called on in Germany to testify for the correctness of a computation: "2 x 6 = 12 according Adam Riese." His books were widely used and imitated for their clear instruction.

St.Albans Book of Computation (1537) *An Introduction for to lerne to Reckon with the Pen or with the counters* after the true cast of Arsemtyke, or Awgrym.

Gemma Rainer Frisius (1508 - 1555) a Dutch physician wrote *Arithmeticae Practicae Methodus facilis* which appeared first in 1540 and with over 60 editions was the most popular treatise in Latin during the 16th century.

Robert Recorde (1510 - 1558) *The Grounde of Artses Teaching the World and Practice of Arithmetik.* A very popular and widely read book in England. Appeared first about 1541 and in 27 further editions up to 1699. He introduced two parallel lines ===== as the symbol for equality, because "no two things could be more equal." He published in 1557 *The Whetstone of Witte*, the first English algebra. The title *Whetstone of Witte*, evidently was a play on the word 'coss', for cos is Latin for whetstone, and the book is devoted to the 'cossike practice', that is algebra.

Jacob Köbel (1470 - 1533) was city clerk in Oppenheim wrote *Rechenbiechlin* in 1514 and *Rechenbuch auff Linien und Ziffern* in 1584 in the mother tongue.

The 16th century arithmetic books all dealt with a number of subjects which will be dealt in the next sections.

8.17.1 Numeratio, an explanation of the Hindu-Arabic numerals and the decimal place-value system. There was much attention to the new zero. The zero was unknown and needless both with Roman numerals and on the counting board.

The switch from the additive Roman numerals to the positional system of the Hindu-Arabic numerals came slowly over a period of centuries. It was not easy for the Late Middle Agers to comprehend the use of the zero which, representing nothing itself, could magically make other digits dramatically increase in value. In 1600 a Dutch source wrote:

0 die men 'nullo' – dat is gheen – noemt, gheen ghetal ofte niet van haar selven, maer een behulp der ander ghetallen ende om te vollen de ledighe plaetse daer gheen ander figuer en comt te staen.

0 called '*nulla figura'* – figure of nothing –, no number by itself, but an aid to other numbers and to fill up the empty places where no figure is needed.

8.17.2 Addition

The operations of addition and subtraction are so elementary that much variation in procedure from early times to the present would not be expected.

Early methods for the basic operations of arithmetic tended to follow patterns suitable for use on a dust board, even when done on paper. The successive stages of an addition of, for example 826 and 483, are represented graphically, as follows (in an addition of Sacrobosco).

826	829	909	1309
483	48	4	

On a dust board or slate the figures would not be copied over as they are here, but rather be erased. It was only until the appearance of printed arithmetic books in the15th century that addition assumed its modern form.

Additions with casting out *the nines*, to check the result:

578	5 + 7 + 8 = 20 – 2 x 9 = 2	add all figures of all numbers and subtract as many
402	4 + 0 + 2 = 6	multiples of 9 as possible. A figure < 9 (digit)
396	3 + 9 + 6 = 18 – 2 x 9 = 0	remains. If the sum of these digits above
	8	the line is equal to the digit beneath the line,
1376	1 + 3 + 7 + 6 = 17 – 9 = 8	the addition probably is carried out well.

The German Johann Widman was the first in 1490 to use the signs + and – in a text. These symbols had been in use as warehouse marks to indicate exes (+) and deficiency (-). The + sign is a ligature for *et*, meaning *and*; it is somewhat similar to an ampersand, &. The – sign may be a simplification of $\overline{m}$ which was previously used to indicate minus in other writings.

8.17.3 Subtraction

Subtraction in the early algorism reveals also peculiarities with the procedures on a counting board. The remainder replaces the minuend, both minuend and subtrahend disappearing.

Fibonacci added the number that was borrowed to the bottom digit in the next column:

6354	8 from 14 is 6 (7 + 1 =) 8 from 15 is 7,
- 2978	10 from 13 is 3 and finally 3 from 6 is 3.
3376	

The Austrian method. The procedure is as follows:

826 Think off the number which added to 483 will give 826; 3 is added to 3 gives 6;

- 483 — 4 added to 8 gives 12; write down 4 and mentally carry 1 to the next 4, making 5;
343 — 3 added to 5 makes 8. 343 is the number which added to 483 gives 826.

8.17.4 Doubling and halving

An interesting method of multiplication related to the Egyptian method is of doubling and halving. These operations were first introduced as separate, independent procedures by al-Khwarismi. Consider 83 x 56. We halve 49, neglecting fractions, and double 28.

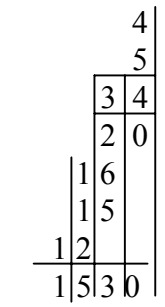
56	28	14	7	3	1	$7 \div 2 = 3\frac{1}{2} \rightarrow 3$ and $3 \div 2 = 1\frac{1}{2} \rightarrow 1$
83	166	332	664	1328	2656	We now add the numbers in the bottom row

under odd numbers in the top row: 664 + 1328 + 2656 = 4648
Each time a remainder of 1 is discarded, the number next to it is added on:
14 x 332 = 7 x 664 = 3 x (2 x 664) + *1 x 664* = 1 x (2 x 1328) + *1 x 1328* + *1 x 664*. These remainders are lost in the progressive halving and must therefor again be gathered together. Every progressive series of halving ends at 1;thus the last double, which is here 1 x 664, must be added in.

8.17.5 Various methods in Multiplication [3)]

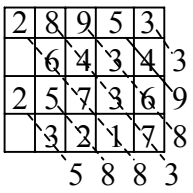
Most arithmetic books start with a square diagram with the tables of multiplication from 10 x 10 to 13 x 13 with the advice to learn this by heart: *Dese tafel moet men leeren alzo vast als uwen Ave Maria zonder iets te messen.* You have to learn this table as good as your Ave Maria without leaving out anything.

From a manuscript dated 1424 an old form of a multiplication is illustrated by the following example of 34 x 45. Here the figures of 45 are written above each other. First multiply 4 x 5, than 4 x 4(0) = 16(0) etc., all in a diagram:

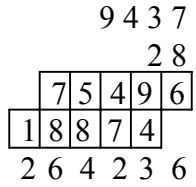


In de *Summa de Arithmetica, Geometrica et Proportionalita* (1494) of Luca di Borgo or Luca Pacioli (1445 - 1514) a multiplication is treated which in Florence was called ***bericocoli*** after an apricot cake, in the example 3217 x 1829: the partial products beneath the former, diagonal added

3217
1829

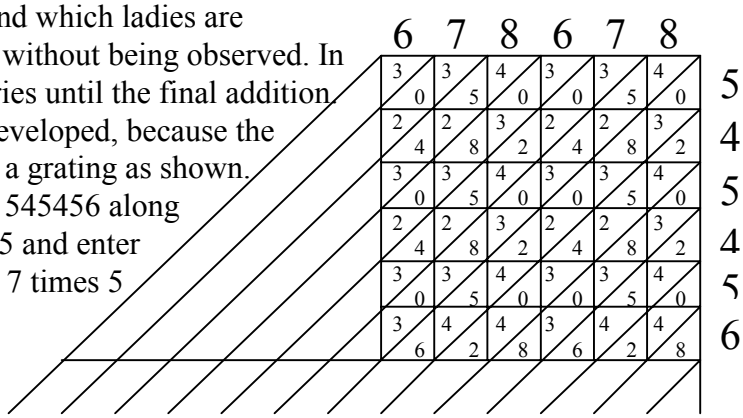


Our method of multiplication was called the ***chessboard*** method by Paciuoli, because its appearance. For example: 9437 x 28



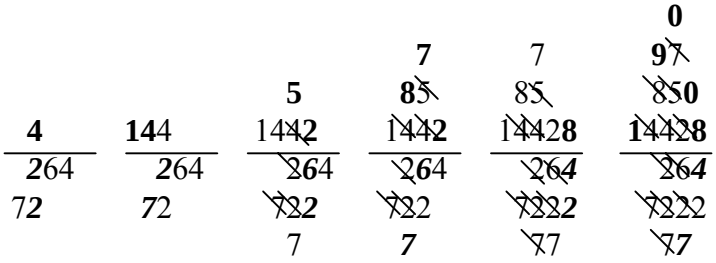
Pacioli introduced the *lattice work* or *jealousy* or ***gelosia*** method.

The jealousy is that screen behind which ladies are accustomed to stand to observe without being observed. In this method one avoids any carries until the final addition. It was used until printing was developed, because the grating was hard to print. Make a grating as shown. Enter 678678 along the top and 545456 along the right side. Multiply 8 times 5 and enter the 40 as shown. Continue with 7 times 5 and so on.



When all the entries have been 3 7 0 1 8 8 9 8 7 1 6 8
made, the numbers along the diagonals are added, starting at the lower right. Carrying is done where needed. The product is 370188987168.

Multiplication with the ***crossing out*** method. The calculation is from a arithmatic book dated 1599: *multiply 72 by 264*, step by step.



The units ***2*** of 72 are placed beneath the highest order ***2*** of 264. Take the ***2*** of 72 and multiply by the ***2*** of 264, writing the **4** above. Now take the ***7*** of 72 and multiply by the ***2*** of 264, writing 14 above and crossing out the 2 of 264.

Then move the 72 one place to the right by writing the digits in the next open space and cross out the initial 72. Continue successively by ***2*** x ***6*** = **12,** ***7*** x ***6*** = **42,** ***2*** x ***4*** = **8** and ***7*** x ***4*** = **28** by writing each succeeding digit in the next open space above and immediately combined the carry-overs and crossing out the finished digits and moving the factor 72 one place to the right. The final product **19008** was read off from the *peak numbers* that were not crossed out.

Complementary multiplication: (1569)

Dese twee Addeert	⎛ *7 3* ⎞	*Dese Multipliceert*	In this method, only the
Ende de 10 achterlaet	⎝ 8 *2* ⎠		tables of multiplication
Comt	*5 6*		up to 5 is needed.

That means, if one has to multiply 7 x 8, than first subtract 10 - 7 = 3 and 10 - 8 = 2, then multiply 3 x 2 = 6, write 6 beneath. Then add 7 + 8 = 15 and ignore the 1 of 10 and write only the 5 in front of the 6, result 56. See also finger-reckoning, chapter 16.

Multiplication with *casting out the nines*, to check the result: example (1600)

2968475
7
20779325

Add all figures of the multiplicand 2 + 9 + 6 etc. Subtract as many multiples of 9 as possible. Place the remainder 5 in top of the cross. Do the same with the result, write the remainder 8 at the left side of the cross and the remainder of the multiplier in the bottom of the cross. Multiply 7 x 5 = 35, 3 + 5 = 8, write 8 at the right side of the cross. Because the two digits at the left and the right are the same (8) the multiplication probably is carried out well.

Pacioli gave also the multiplication ***by the cross*** – *multiplicare per crocetta* – which was the method normally used in Venice. This system is only useful for small problems as far as two four-place factors. Example 34 x 25. The thought process involved is illustrated by the following steps, following the lines of the cross. 5 x 4 = 20; put down 0 and carry the 2; 5 x 3 = 15; plus 2 x 4 = 8 is 23 + 2(carried) = 25; put down the 5, carry 2. Then 3 x 2 = 6, plus 2(carried) is 8. Thus the result is 850. Cross multiplication allows the problem to be calculated mentally with only the answer being written down on paper. This procedure also was known by the *lightning method*. Adam Riese remarked in regard to it: *Sie nimmt viel Kopffs* "It takes much head." It was also shown in the *Bamberger Rechenbuch*.

Multiplication ***by factors*** – *multiplicare per repiego*. A divisible number (non prime) 23 x 14 in Italian is *un numero di riprego*. Problem: 23 x 14 = 322; the multiplication x 7 = 161

is done not by the multiplier 14, but by its factors 2 and 7. (23 x 7 x 2) x 2 = 28

result 322

8.17.6 Division

In division by striking-out or crossing-out operation, instead of taking away the finished figures again and again, as happens on the counting board, the writer has to cross them out as in the example below. This method for division was used in India at least by the 12th century and from there it seems to have been carried to Arabia. From the Arabs it passed over to Italy in the 14th century. In 1584 Luca Pacioli called it *divisione vel batello* or *vel Galea* because a computation of this kind in its final result looked like a boat or a ***galley*** with its sail spread. This method lost favour with the advent of printing, but it was the most common method taught until 1600.

Omme te deelen 27648 doer 36…settet 3 onder 7 ende 6 onder 6… Neempt die 3 soe menichmael in 27 datter so veele restet omme te hebben die 6 ook so menichmael. (1567)
To divide 27648 by 36…place 3 under 7 and 6 under 6…Take the 3 as many times in 27 that as much are left to multiply that 6 the same times.

			2	2	~~2~~
	2	2**6**	~~2~~6	~~2~~6**4**	~~2~~64
6	6**4**	~~6~~4	~~6~~4**8**	~~6~~48	~~6~~48
27648 | ***7***	~~2~~7648 | ***7***	~~27~~648 | 7***6***	~~27~~648 | 7***6***	~~27~~648 | 76***8***	~~27~~648 | 76***8***
3*6*	***3****6*	~~3~~6*6*	~~3~~6***6***	~~3~~6~~6~~*6*	~~3~~6~~6~~***6***
		3	*3*	~~3~~***3***	~~3~~*3*

7 x ***3*** = 21 27 - 21 = **6** ***6*** x ***3*** = 18 24 -18 = **6** ***8*** x ***3*** = 24 28 -24 = **4**
7 x ***6*** = 42 66 - 42 = **24** ***6*** x ***6*** = 36 van 64 = **28** ***8*** x ***6*** = 48 48 - 48 = 0

The divisor was placed beneath and the remainder was written above the dividend and in the column where a free space was available. The same is true of the digits 3 and 6 of the divisor 36 moving to the right. The digits were crossed out as they were finished with. Cross-out or erase the intermediate steps was still in use in books up to 1584.

A Greek division method related to the galley method in that way that the divisor is multiplied digit-by-digit. Consider the problem 625 ÷ 25

25)	625 (2	Divide 25 into 62, giving 2.
	4	Multiplying, 2 x 2 (of 25) = 4.
	22	Subtract the 4 from the 6 and bring down the 2, giving 22.
	10	Now, 2 times 5 (of 25) is 10, and subtracting gives 12.
	125	Bring down the 5, and divide 25 into 125, giving 5.

25)	625 (25	
	4	
	22	
	10	
	125	Multiplying 5 x 2 = 10.
	10	Subtract the 10 from the 12

—

25	and bring down the 5, giving 25.
25	Now, 5 times 5 is 25, and subtracting gives 0
0	The quotient is 25.

8.17.7 Extracting square roots

Extracting square roots is only treated in a few books because this is not a 'specie', that means it is no basic calculation as adding, subtracting, multiplication and division.

Example: $\sqrt{481636}$ (1569)

12 48 | 16 | 36 ***6*** *12*	Divide the number 481636, from the right-hand side, into pairs of two figures by small lines. Select the first figure of the root in such a manner that the square is as large as possible but < 48, → ***6***. Write this 6 beneath 48 between the lines and 2 x ***6*** = *12* beneath the lines. 48 - $\boldsymbol{6}^2$ = 12, write this 12 above 48 and cross out 48. The 2nd figure of the root in 2 steps:
12 ~~48~~ | 16 | 36 ***6 9*** 12	First step: select he second figure *x* in such a manner that the difference 121 - 2.***6***.*x* - x^2 is as small as possible: → ***9***: write this ***9*** between the lines. 121- 12.***9*** = 121-108 = 13.
1 ~~12~~ 3 ~~48~~ | ~~16~~ | 36 ***6 9*** 12	Write 13 above 121 and cross out this 121. Second step: 136 - $\boldsymbol{9}^2$ = 136 - 81 = 55. (in one step: 1216 - 2.6.9.10 - 9^2 = 55)
1 5 ~~12~~ 35 ~~48~~ | ~~16~~ | 36 ***6 9*** ~~12~~ *38* *1*	Write 55 above 136 and cross out this 136. Cross out 12 beneath the line. Write 2.***69*** = *138* beneath the line.
~~1~~ 5 ~~12~~ ~~35~~ 1 ~~48~~ | ~~16~~ | 36 ***6 9 4*** ~~12~~ *38* *1*	For the 3rd figure of the root, take *x* = ***4,*** write the third figure between the lines. First step: 553 - *138*.***4*** = 553 - 552 = 1. Second step: 16 - 4^2 = 0. (in one step: 5536 - 2.69.4.10 - 4^2 = 0) The square root of 481636 = 694.

8.18 The gradual penetration of the Hindu-Arabic numerals [4)]

The fact that the Roman numerals were so deeply rooted in the customs and affections of the people at first made it exceedingly difficult for the new Hindu-Arabic numerals, the *figure numbers*, to replace the old familiar Roman numerals. The Hindu-Arabic numerals first began to be used by ordinary people about 1500.
Examples of the resistance against the introducing of the new numerals:
- the prohibition in the regulations of the Arte del Cambio (the guild of money-changers) in Florence in 1299, to use the new Hindu-Arabic numerals in the cash-books.

- only after 1494 only Hindu-Arabic numerals occurred in the financial columns of the business books of all merchants of the Medici-family;
- even as late as 1594 the merchants of Antwerp were warned not to use Hindu-Arabic numerals in contracts or bank drafts.

The Hindu-Arabic numerals were very easy to alter or forge, for example, changing 0 to a 6 or 9 would have been quite easy. The merchants had developed various tricks to prevent this sort of things with Roman numerals; for example XII would be written as Xij so that an extra i could not be added to the end and by marking the first character a capital, they prevented anyone from adding characters to the left-hand side of a number.

Examples of these Roman numerals:

1) From the Treasury Account Books of 1540 (Temp Henry XIII):
   Lxx li. Xiij s. Viij d., which is £70 13s. 8d.
2) Used in a book of receipts by an official of Rüsselheim, 1554. The result reads:
   Vj^{M} Vij^{C} xiij Gulden Xv Heller = 6713 Gulden and 15 Heller.

8.19 Counter numerals

In England there was a different special way of placing and indicating the value of counters on the board, where during the 16th century it came into common use in government offices (along with the line board). (See Robert Recorde in *Ground of artes Teaching the Worke and Practice of Arithmetik*).

This used a board with horizontal bars, which in turn were divided into individual vertical fields, for *£20, £1, s,* and *d* respectively.

This counting board has no provision for decimal or quinary grouping; these were very cleverly introduced by special arrangements of the counters themselves. If a single counter ○ was placed above another at the left end of the field, it meant 10; likewise a counter placed above at the right end meant 5, except in the *d* field, where it stood for 6.

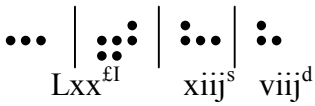
Score of £	£	*s*	*d*
○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○ ○	○ ○ ○ ○ ○
100	10 5	10 5	6
20 20 20	1 1 1	1 1 1	1 1 1
20	1	1	1 1

A score of £ is £ 20. The coefficient for conversion from £ 20 to £ 1 and from £ 1 to *s* is 20, but from *s* to *d* it is 12.

Three score and 1 = 61

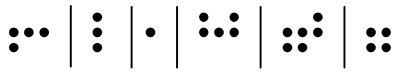
Through this place-value arrangement, the counters actually formed true number symbols. These 'counter numerals' were actually taken directly from the counting board and were written down on paper until well into the 17th century.

In the Treasury Account of 1540 amounts are written in Roman numerals, the *dot-diagram* shows this amount also as indicated by the counters lying on the table at the end of the operation:



In a Exchequer Augmentation Office Book from 1600 the amounts were still written in Roman numerals but besides the 'dot-diagram' the total in Hindu-Arabic numerals.

The appearance of a 'dot-diagram' in the Exchequer record books in 1676 is convincing evidence of the



use of counters even after Arabic notation had been introduced: 4438£ 10s 4p • 4000 400 20 18 10 s 4 p

Performing calculation with counters along ‘calculation with the pen’ existed for a long time. In 1707 Leonard Sturm still treated calculation with counters in his *Kurzer Begriff der gesamten Mathesis*.

At the time of the French Revolution (1789) the use of jetons (counters) and boards in calculation at schools and at the authorities was banned.
The administration of Basel still calculated on the counting board up to 1798.

Luther (1483 - 1546), who often drew on the counting board for figures of speech in his sermons, is an unimpeachable witness for the common people’s daily familiarity with the reckoning board.

Goethe (1749 - 1832), in a passage from a letter by Bettina von Arnim says: "No toy fascinated him more than his father’s counting board, on which he would reproduce the constellations of the stars with counters".

8.20 Astronomy

By the 15th century astronomers were having very serious difficulties accepting Ptolemy’s system in detail. As European explorers set out on voyages around the globe, they found so many parts of the world previously unknown to them that they realised that Ptolemy’s *Geografica* was in error. The way was prepared for believing that the fundamentals of his astronomy could be wrong. Regiomontanus pointed out one type of error.

8.20.1 Regiomontanus [4)]

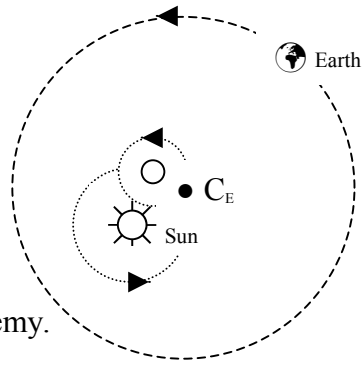
Johannes Müller (1436 - 1476), was called **Regiomontanus** from his birthplace, Königsberg, (King’s mountain), in Franconia. Having studied at the universities of Leipzig and Vienna, where he developed a love for mathematics and astronomy, he accompanied Cardinal Bessarion to Rome, where he acquired a proficiency in Greek. After this time, he returned to Germany and settled at Nuremberg.
In astronomy his chief contribution was the completion of a new Latin version of Ptolemy’s *Almagest*. Of greater significance for mathematics, however, was his *De triangulis omnimodis libri quinque* – On triangles of all Kinds (finished in 1464), a systematic account of the methods for solving triangles which marked the rebirth of trigonometry. This treatise remained in manuscript until 1533. The book reprinted at Basel in 1561, was widely studied during the sixteenth century. In 1475 he had been invited to Rome by Pope Sixtus IV to share in one of the perennial attempts to reform the calendar, but he died there shortly after he had arrived.

8.20.2 Nicolaus Copernicus and the heliocentric system [5)]

Nicolaus Copernicus (1473 – 1543), was born in Torun in West Prussia (now Poland), a copper-mining town. Copernicus is the Latin form of his Polish name, Koppernick. He was sent at the age of 18 to study at the University of Cracow. After this time he was permitted to travel to Italy at Bologna and Padua over the next ten years. He studied Ptolemy’s system in great detail and became aware of all its inaccuracies. He came to the conclusion that it was impossible to patch up the earth-centred approach any longer. Copernicus decided to read all

the opinions of the ancients to determine whether anyone had proposed a system of the universe different from the earth-centred one. Having discovered that Aristarchus of Samos had proposed a heliocentric system in which the earth moves, Copernicus explored the consequences of reforming the system. Copernicus returned to Poland in 1510. Copernicus' fundamental treatise in which he expounded his system of the universe was *De Revolutionibus Orbium Coelestium,* On the revolution of the Heavenly Spheres, a book that represented the work of a lifetime but that was only published in 1530, the year of his death.

It should be emphasised that Copernicus, like his predecessors, conceived of the system of the universe as a series of nested spheres containing the planets, rather than as a empty space through which the planets travel in circles. Copernicus had no physics to keep the planets in their orbits. One of Copernicus's aims in his reform of Ptolemy's work was to return to one of the classic principles of astronomy, that all heavenly motion must be composed of uniform motion of circles about their centres. Copernicus believed Ptolemy had violated that principle by accounting for certain aspects of a planet's motion through the use of the equant, a point within the planet's orbit around which the radius vector to the centre of the planet's epicycle revolved uniformly. Copernicus found that simply placing the planet's on sun-centred spheres did not satisfy the requirements of observation.
Thus he, like Ptolemy, had to introduce more complicated devices such as epicycles. Copernicus's calculations showed that the centre of the earth's circular orbit was not the sun, but a point C_E in space that revolved on a epicycle whose centre O revolved about the sun. Similarly, the centres of the various planetary orbits were neither located at the sun nor even at the centre of earth's orbit. In the end, the full system as described in *De Revolutionibus* was of the same order of complexity as that of Ptolemy.



The Lutheran theologian Andreas Osiander, who wrote the preface to the printed text of *De Revolutionibus,* claimed that Copernicus' views on the earth's motion should not taken as true, but only as a hypothesis for calculation, "since the true laws cannot be reached by use of reason." The Catholic Church had little to say about Copernicus' work. In fact, it was taught at various Catholic universities, and the astronomical tables derived from it provided the basis for the reform of the calendar promulgated by Pope Gregory XIII in 1582.

8.20.3 Tycho Brahe [5)]

Tycho Brahe (1546 – 1601) was an astronomer who devoted much of his life to making better observations to improve astronomical tables. In 1576 he was able to convince King Frederick II of Denmark to allow him the use of the island of H'veen near Copenhagen and to provide him with funds for building a magnificent observatory.
Two of Tycho's most important series of observations convinced him that the Ptolemaic system could not be correct. First, beginning in late 1572, he tracked for 16 months a new object, which had appeared in the heavens, a *nova*. Despite Aristotle, change was possible in the heavens. The possibility of change in the heavens was further confirmed by his observation of a *comet* in 1577. By a comparison of the parallax of the comet with that of the moon and the planets, he concluded that the comet lay beyond the moon and that it revolved around the sun at a distance greater than that of Venus. Since its distance from the sun apparently varied greatly during the course of his observations, Tycho concluded that the

heavens could not be filled by solid spheres carrying the planets. There must in fact be space between the planets in which another heavenly object could travel.
The last two years he worked together with Johann Kepler in Prague.

8.20.4 J ohann Kepler[5)]

Johann Kepler (1571 – 1630), is best known as an astronomer. He was a professor at Graz from 1593 to 1599, assistant to Tycho Brahe for the final two years of Brahe's life in Prague , where, following Brahe's death, he became mathematician to the Emperor Rudolph II. Kepler was able to use the mass of Tycho's observations to construct a new heliocentric theory, which could accurately predict heavenly events without the elaborate machinery of epicycles. From 1612 to 1626 he taught at Linz.

In his *Mysterium Cosmographicum*, Secret of the Universe (1556), Kepler discussed the numerical relationship with which God had created the world. Why are there precisely six planets?
Because 'God is always a Geometer,' the Supreme Mathematician wanted to separate the planets with the regular solids. Euclid had proved that there could be only five such solids, so Kepler took this as the reason that God chose to provide just six planets. He then worked out the idea that between each pair of spheres containing the orbits of adjacent planets there was inscribed one of the regular solids. Thus, inside the sphere of Saturn was to be inscribe a cube (6 planes) or hexahedron, which in turn circumscribed the orbit of Jupiter. Similarly, between the orbits of Jupiter and Mars was a pyramid (4 planes) or tetrahedron, between Mars and Earth a dodecahedron (20 planes), between Earth and Venus an icosahedron (12 planes), and finally between Venus and Mercury an octahedron (8 planes). These solids lay in the inter spherical spaces, and their sizes provided a measure of the relationship between the sizes of the various planetary orbits.

In 1600 Kepler visited Henry Briggs for instruction about the theory of ellipses.
As early as 1604 Kepler had become involved with conic sections through work in optics and the properties of parabolic mirrors. With a strong imagination and a Pythagorean feeling for mathematical harmony, Kepler developed for conics in 1604 (in his *Ad Vitellionem paralipomena*, that is *Introduction to Vitello's Optics*) what we call the principle of continuity. From the conic section made up simply of two intersecting lines, in which the two foci coincide at the point of intersection, we pass gradually through infinitely many hyperbolas as one focus moves farther and farther from the other. When the one focus is infinitely far away, we no longer have the doublebranched hyperbola, but the parabola. As the moving focus passes beyond infinity and approaches again the other side, we pass through infinitely many ellipses until, when the foci again coincide, we reach the circle. The idea that parabola has two foci, one at infinity, is due to Kepler, as is also the word 'focus' (Latin for 'hearth side').

In his *Astronomia nova* (Heidelberg, 1609) he announced his first two laws of astronomy:
1) Planets move in elliptical orbits about the sun with the sun at one focus, and
2) The radius vector joining a planet to the sun sweeps out equal areas in equal times.

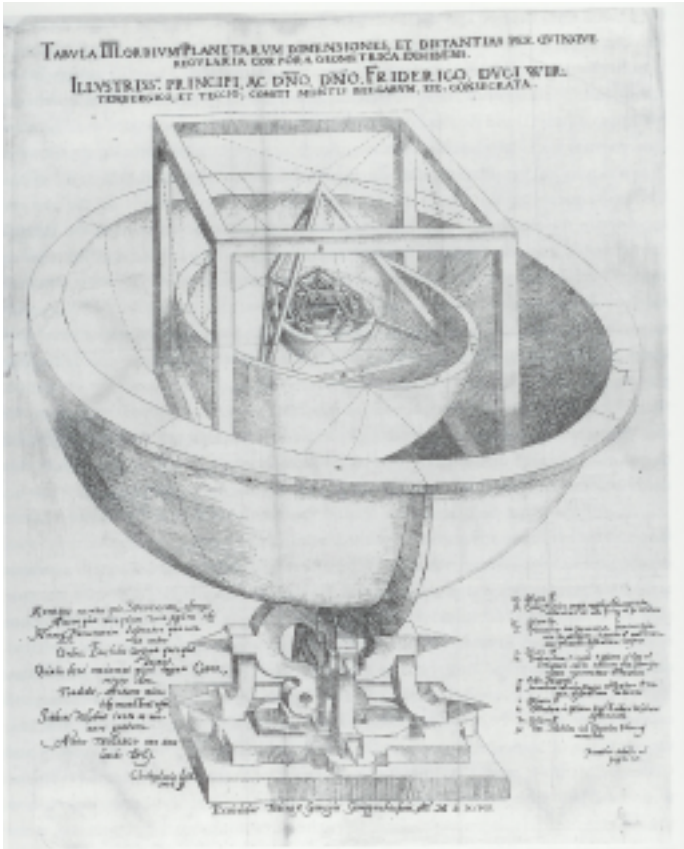
Kepler's third law appears for the first time in 1618 in the *Harmonices Mundi* stated as an empirical fact: "It is absolutely certain and exact that the ratio which exists between the

periodic times of any two planets is precisely the ratio of the 3/2th power of the mean distances (of the planet to the sun)" or

3) The squares of periods of revolution are to each other as the cubes of the semimajor axes: $\frac{T_1^2}{T_2^2} = \frac{R_1^3}{R_2^3}$

Imagine the amount of computation Kepler must have done in order to discover the last law. He and all other astronomers impatiently awaited the development of a simpler method of especially multiplication and division.

In the next chapter follows the laborious but successfully way to logarithms.



1) Karl Menninger
Number Words ans Number Symbols

2) Marjolein Kool: Die conste vanden getale
A book (thesis) completely dedicated to the Dutch arithmetic books and arithmetic methods from the 15th and 16th century.

3) Arthur Gittleman
History of Mathematics

4) Boyer & Merzbach
A History of Mathematics

5) Victor J. Katz
A History of Mathematics

Chapter nine, *The invention of Logarithms,*

9.1 Exponential and logarithmic functions [1)]

To denote five 2's multiplied together, we can write 2^5, so that $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$.
If a is any positive real number, and m is any positive integer, a^m means a multiplied by itself m times. (Positive means > 0 and integer means a hole number)
If m and n are any positive integers, then $a^m \times a^n = a^{m+n}$ (since each side denotes the number a multiplied by itself $m + n$ times. If we put $n = 0$ in above equation, we would have $a^m \times a^0 = a^{m+0} = a^m$ and we require that $a^0 = 1$.
The function a^x is called an exponential function, and x is called the exponent.

We can define $y = a^x$ for any fixed choice of $a > 0$ and for all real values of x.
To each value of x we have a unique value of y, so that y is a function of x. But it is also true that given any positive real value of y, there is a unique real value of x. This means that x is a function of y. It is called a logarithmic function, and to emphasise its dependence on the number a, we say that x is the logarithm of y to the base a; we write $x = {}^a\log y$.
For any positive real a, the corresponding exponential and logarithmic functions $y = a^x$ and $x = {}^a\log y$ are said to be inverse functions of each other.
Given two real numbers x and z and let us define $y_1 = a^x$ and $y_2 = a^z$ for some fixed choice of $a > 0$, than $y_1y_2 = a^{x+z}$, and so we have $x = {}^a\log y_1$ and $z = {}^a\log y_2$ and ${}^a\log y_1y_2 = x + z$ and ${}^a\log y_1y_2 = {}^a\log y_1 + {}^a\log y_2$: a multiplication is convert into an addition!
Given that we usually express our numbers in the decimal system, 10 is a more practical base:
${}^{10}\log 5612 = {}^{10}\log(1000 \times 5{,}612) = {}^{10}\log(10^3 \times 5{,}612) = {}^{10}\log 10^3 + {}^{10}\log 5{,}612 = 3 + {}^{10}\log 5{,}612$
${}^{10}\log 0.05612 = {}^{10}\log(0.01 \times 5{,}612) = {}^{10}\log 10^{-2} + {}^{10}\log 5{,}612 = -2 + {}^{10}\log 5{,}612$
Thus we do not need to tabulate values of base 10 logarithms outside the range [1,10].

Examples: ${}^{10}\log 100 = 2$ for $10^2 = 100$ and ${}^{10}\log 1000 = 3$ for $10^3 = 1000$ and so on.
Also ${}^{10}\log 10 = 1$ ${}^{10}\log 1 = 0$ and ${}^{10}\log 0{,}1 = -1$
The question was for a long time: what are the logarithms of the interjacent numbers?

9.2 Archimedes [2)]

In a letter to King Hiëro II of Syracuse, Archimedes (287 to 212 B.C.) wrote already:
"If there be any number of terms of a series in continued proportion, say (in modern numbers) $A_1, A_2, A_3, \ldots A_n, \ldots A_m, \ldots A_{m+n-1}, \ldots$ of which $A_1 = 1$, $A_2 = 10$ [so that the series forms the geometrical progression $1, 10^1, 10^2, \ldots 10^{m-1}, \ldots 10^{n-1}, \ldots 10^{m+n-2}, \ldots$], and if any two terms as A_m, A_n be taken and multiplied, the product $A_m \times A_n$ will be a term in the same series and will be as many terms distant from A_n as A_m is distant from A_1; also it will be distant from A_1 by a number of terms less by one than the sum of the numbers of terms by which A_m and A_n respectively are distant from A_1".

Recorded in our figures and letters in general form:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	number of terms
1	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	a^{12}	a^{13}	geometrical series

According Archimedes w see $a^3 \times a^8 = a^{11}$ and for the number of terms $4 + 9 - 1 = 12$.
Would Archimedes have known our way of notation than he possible would have discovered the base of logarithm?

9.3 Nicolas Chuquet [3)]

Nicolas Chuquet (ca. 1430 to ca. 1487), born at Paris, took his bachelor's degree in medicine, and practised at Lyons. In 1484 a manuscript was composed, written in the vernacular, entitled:

Triparty en la science des nombres.

The first of the 'Three Parts' concerns the rational arithmetic operations on numbers, including an explanation of the Hindu–Arabic numerals.
The four fundamental operations being indicated by the words and phrases *plus, moins, multiplier par* and *partyr par*, the first two sometimes abbreviated in the medieval manner as $\overline{p}$ and $\overline{m}$.

In the second part, concerning roots of numbers the modern expression: $\sqrt{14-\sqrt{180}}$ appears in the not very dissimilar from R)^2.14 .$\overline{m}$. R)^{2}180 .
The second power he called *champs*, the third power *cubiez*, and the fourth *champs de champ,* or: nombres seconds, tiers, and quarts.
For multiples of these Chuquet invented an exponential notation of great significance. The denominacion or power of the unknown quantity was indicated by an exponent associated with the coefficient of the term, so that our modern expressions in the *Triparty* appeared as:

$5x = .5.^{1}$ $\quad 6x^2 = .6.^{2}$ $\quad 9x^0 = .9.^{0}$ $\quad -12^{-2} = \overline{m}12^{2\overline{m}} =$ and

$9x^{-2} = .9.^{2\overline{m}} =$ .9.seconds moins and $\dfrac{.72.^{1}}{.8.^{3}} = .9.^{2\overline{m}}$ or $\dfrac{72x}{8x^3} = 9x^{-2}$

Related to these laws is his observation of the relationship between the powers of the number two, and the indices of these powers set out in a table from 0 to 20, in which sums of the indices correspond to products of the powers:

0	1	2	3	4	5	6	7	8 .	 20	= denominacion or denominator
2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	20^{20}	
1	2	4	8	16	32	64	128	256 . .	1.048.576 . .	

He wrote: "The product of 2 numbers of the lower line is a number of the same line and its denominator is equal to the sum of the denominators of the numbers".
Except for the magnitude of the gaps between entries, this constituted a miniature table of logarithms to the base two.
Again a step closer!

The *Triparty* was not printed until 1880, and probably was known to a few mathematicians.

9.4 Michael Stifel [4 & 5)]

Michael Stifel (1487 – 1567) was ordained as a priest in 1511. Reacting to various clerical abuses he became an early follower of Martin Luther. In the 1520s he became interested in what he called *wortrechnung* (word calculus), the interpretation of words through the numerical values of the letters involved. Through interpreting certain Biblical passages using his numerical methods, he finally came to the belief that the world would end on October 18, 1533. He assembled his congregation in the church on that morning, but to his great dismay, nothing happened, He was subsequently discharged from his parish and for a time placed

under house arrest. Because he had now been cured of prophesying, however, he was given another parish in 1535 through the intervention of Luther. Subsequently he devoted himself to the study of mathematics at the University of Wittenberg and soon became an expert in algebraic methods. In 1544 he published his *Arithmetica Integra*, in which he wrote out a table of powers of 2 (geometrical progression) alone with their indices, which included the negative values -1, -2, -3, -4 as corresponding to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ respectively:

...	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	indices
...	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	256	512	geometrical progression

He formulated the relation between the terms of a geometrical progression and the corresponding indices of the common ratio as follows: "If we multiply any two terms of the progression, the result would be the same as if we had *added* the corresponding indices. For example, 4 x 16 = 64, a result that could have been obtained by adding the indices 2 and 4. Similarly, dividing one term of a geometric progression by another term is equivalent to *subtracting* their indices. Doubling of the indices agree with the squares of the powers, reversely, halving the indices agree with the roots of the powers. A problem arises, however, if the indice of the denominator is greater than that of the numerator, as in 8 divided by 32; our rule would give 3 – 5 = -2. Now we simply define ¼ = -2. Now we can extend a geometric progression indefinitely in both directions".
With this, Stifel has approximated closer the theory of logarithms. The development of a method to calculate interjacent values of the logarithms still remained. Stifel did not know exponents as Chuquet did. In 1545 Stifel published also *Deutsche Arithmetica*.

9.5 J ostBürgi [3,5 & 6)]

Jost Bürgi (1552 – 1632), mathematician, astronomer and famous constructor of astronomical watches, celestial globes, and inventor of among others the proportional compass.
He served to the court of Landgrave Wilhelm IV of Hessen at Kassel and to the court of the uncle of Wilhelm, Kaiser Rudolf II at Prague. There he got acquainted with Kepler.

The idea of logarithms had possible occurred to Bürgi as early as 1588. In that year the astronomer Reimarus Ursus Dihmarsus wrote: "Bürgi possesses a device with which his calculations become extremely easy." With this he aimed undoubtedly at the logarithms, which Bürgi published at Prague only in 1620 after long insisting of Kepler:

Aritmetische und Geometrische Progreß Tabulen.

It was only an anonymous printing of a short table of what we would call antilogarithms. Because the lac of a manual and/or example calculations his tables were for other people unserviceable or unfit for use and probably were thrown away. It is a wonder that two copies were saved. Behind the copy from Danzig probably used by Bürgi himself a hand written manual is attached, that only was described in 1856. In this manual also the sequences:

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	4	8	16	32	64	128	256	512	1024	2048	4096

So, Bürgi also proceeded from the properties of arithmetic and geometric series. Instead of proceeding from a number a little *less* than one (like Napier did, see 9.6.2), Bürgi chose a number a little *larger* than one – the number $1 + 10^{-4} = 1{,}0001$ and instead of multiplying powers of this number by 10^7, Bürgi multiplied by 10^8 . He multiplied all of his power indices

by ten in his tabulation. That is, if $N = 10^8(1 + 10^{-4})^L$, Bürgi called $10L$ the 'rote' (red) numbers corresponding to the 'black' number N, after which appeared the next series:

0	0,0001	0,0002	0,0003	0,0004	etc. the numbers 0,0001n and
1	$1{,}0001^1$	$0{,}0001^2$	$1{,}0001^3$	$1{,}0001^4$	the 'black' numbers as $1{,}0001^n$

For the 'black' numbers a continuous multiplication can be performed as is done to the right:

```
1,0001          10001  =  10
              x 10001
                10001
              10001
              100020001  =  20
                x 10001
              100020001
          100020001
          100030003      =  30
            x   10001
          100030003
      100030003
      100040006          =  40
          and so on.
```

It can easier be done with only addition as is done below:

```
100 000 000     in the strict sense 1,000
  +  10 000     this is removed 10^4
100 010 000
  +  10 001
100 020 001
  +  10 002
100 030 003
  +  10 003
100 040 006
  +  10 004
100 050 010
```

After that Bürgi could make his table from which a part of page 1 is shown in Figure 9.1.

	0	500	3500
0	100000000	100501227	10356179
10	 10000	 11277	 7214
20	 20001	 21328	 8250
30	 30003	 31380	 9286
40	 40006	 41433	10360322
50	 50010	 51487	 1358
60	 60015	 61543	 2394
70	 70021	71599	 3430
80	 80028	81656	 4466
90	 90036	91714	 5503
100	100100045	100601773	 6539
110	 10055	11834	 7576
120	 10066	 21895	 8613
130	 10078	31957	 9650
140	 10091	42020	10370687
.....	.....	.....	.....
500	100501227	101004966	10408051

Figure 9.1

In publishing his table, Bürgi placed his red numbers on the side of the page and his black numbers in the body of the table; hence he had what we should describe as an antilogarithm table. The red numbers are the logarithms of the black numbers. Example:

$x = 1{,}0005001 \times 1{,}00060015$
$\log x = \log 1{,}0005001 + \log 1{,}00060015 =$
$50 + 60 = 110$
From this follows: $x = 1{,}00110055$

In one respect Bürgi's logarithms come closer to ours than do Napier's, for as Bürgi's black numbers increase, so do the red numbers.

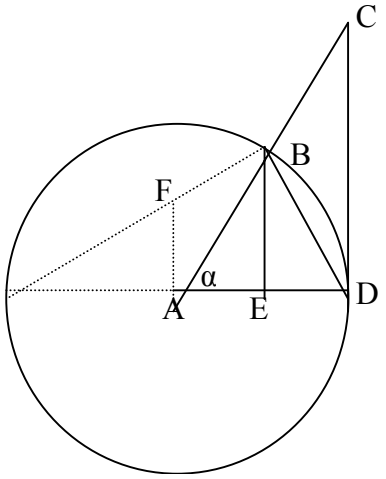
Jost Bürgi was very modest and signed only with his initials J.B. He had the intention to write a complete work on mathematics in which his logarithms should be inserted.

Bürgi must be regarded as an independent discoverer who lost credit for the invention because of Napier's priority in publication.

9.6 J ohn Napier (1550 to 1617)[3 & 7)]

The son of Sir Archibald Napier and his first wife, Janet Bothwell, John, was born in 1550 at his family's estate, Merchiston Castle, near Edinburgh, Scotland. In 1563 John's mother died, and then John was sent off to St. Salvator's College at the University of St. Andrews, where he studied Latin and the other subjects then taught at St. Andrews. After the death of Sir Archibald in 1608, John returned to Merchiston, where, as the eight laird of the castle, he spent the rest of his life. Napier was not a professional mathematician but in the awakened interest in astronomy and in the attendant development of trigonometry, generated also by other forms of world exploration – land surveying, cartography, and navigation, he occupied himself much among others with trigonometrically calculations.

In Napier's time, trigonometric functions were not considered by the ratios of the lengths of the sides of a right-angled triangle, but were considered as the lengths of various lines drawn in and around a circle of some convenient radius, see Figure 9.2.



sine of α was the length of the line BE

tangent of α was the length of the line CD

secant of α was the length of the line AC

versed sine = 1-cos α = ED

half tangent of α was the length of the line AF (really the tangent of half the angle)

chord of α was the length of the line BD

Figure 9.2

Of course, the lengths of these lines would change each time a circle of different radius were used, so the mathematician consulting any trigonometric table had to be aware of exactly how long the defining radius was for the tables then in use.

Scientists everywhere began to spend enormous amounts of time in calculating tables of trigonometric functions. Napier voiced his opinion thus: "There is nothing more troublesome in mathematics than the multiplication, division, square and cube root extraction of great numbers which involve a tedious expenditure of time, as well as being subject to slippery errors". It became important to find methods of replacing the often-laborious operations of multiplication and division with addition and subtraction.

In 1590, Dr. John Craig, physician to James IV of Scotland visited his friend, John Napier. Craig had been in the party when James VI in that year had sailed with a delegation for Denmark to meet his bride-to-be, Anne of Denmark. The party had been forced by storms to land not on Copenhagen, but on the shore of the island of H'veen not far away from the observatory of Tycho Brahe. While awaiting more favourable weather, they were entertained by the astronomer. Craig told Napier that Brahe used the *prosthaphaeresis* method, represented by:

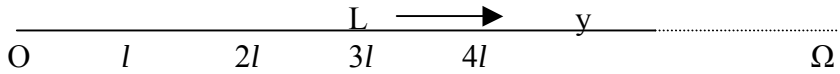
$$\sin \alpha \times \sin \beta = [\cos(\alpha - \beta) - \cos(\alpha + \beta)] / 2$$

(The Arabian mathematician Ibn Junus († 1009) already described this in words.)

Once it had been shown that a rather nastly multiplication could be replaced by a few simple additions, subtractions, and an elementary division by 2, it is entirely likely that this formula spurred mathematicians, including Napier, to search for other methods to simplify the harder arithmetical operations. Because we know that Napier knew of the method of *prosthaphaeresis*, it may well have influenced his thinking because the first logarithms were not logarithms of numbers but logarithms of sines. Another factor in the development of logarithms at this time was that the properties of arithmetic and geometric series had been studied extensively in the previous century.

9.6.2 The idea [8)]

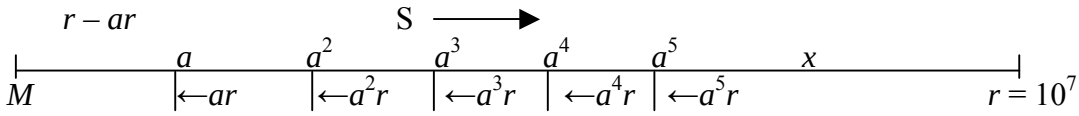
On the search for a method to make steps smaller between the relations of an arithmetical and a geometrical progression, Napier had the luminous thought of a floating scale. He took an infinite line O - Ω and on that line from the starting point O, according to an increasing arithmetical progression, the points l $2l$ $3l$ $4l$ etc., with equal distances:



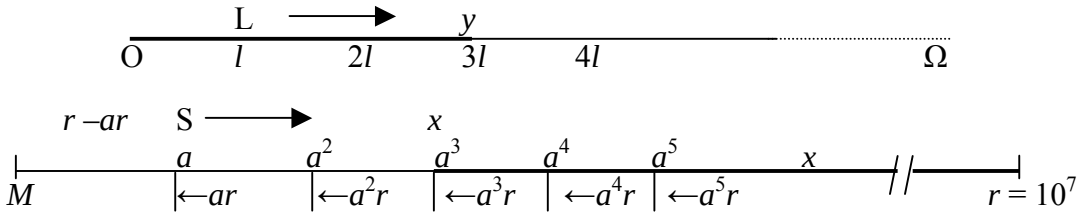
Napier made move a point L on the line O – Ω from O with a fixed initial and constant velocity so that L covers equal distances in equal spaces of time:

O – l l – $2l$ $2l$ – $3l$ $3l$ – $4l$ etc.

After that, Napier took a line segment M – r with a length $r = 10^7$. Although Napier already was familiar with decimals he persisted in the practice to introduce for trigonometrically functions for the radius a large value to avoid fractions. Therefore he could express the values in his table in 7 figures without decimals.
Than Napier took a number a, smaller than 1, but close to 1 ($1 - 10^{-7} = 0{,}9999999$) and placed on the line M – r a number of points from the starting point M according the decreasing geometrical progression a a^2 a^3 a^4 a^5 etc.



The distances from these points to the right-hand end point r also form a decreasing geometrical progression ar a^2r a^3r a^4r a^5r etc.
Furthermore the distances from M are $r - ar$ $r - a^2r$ $r - a^3r$ $r - a^4r$ $r - a^5r$ etc.
Simultaneously with L (on O – Ω), he made start on M – r from M another point S with the same start velocity as from point L, such that S reaches in the same spaces of time as from L the points a a^2 a^3 a^4 a^5 etc., in other words, S moves with a variable speed which decreases proportionally with its distance to r. In general: when L reaches on O – Ω the point y and on the same moment S reaches on M – r the point x, than to every distance O – y on O – Ω belongs on M – r the remaining part x – r and Napier called the distance O – y the logarithm of the distance x – r (the logarithm of a given sine x).



Thus the logarithm table of Napier has no relation to the logarithms to numbers but instead to sines and tangents of angles.

Napier chose 10^7 as a value for the radius of the circle. As base for his powers, he decided on $(1-10^{-7}) = 0.999\ 999\ 9$. By multiplying all powers by 10^7, we have the relation $N = 10^7(1-10^{-7})^L$, where L is the 'Neperian logarithm' of the number N.

By dividing Neperian logarithms and their corresponding numbers (antilogarithms) by 10^7, we nearly obtain a system of logarithms to the base of $\frac{1}{e}$, because:

$$\left(1 - \frac{1}{10^7}\right)^{10^7} \approx \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$$

The natural logarithm = l_n, where ${}^e\log a = l_n a$ $\quad e = \lim_{n\to\infty}(1 + \frac{1}{n})^n = 2{,}718281828459\ldots$
The Neperian logarithm Nep.log$x = 10^7.l_n\ 10^7/x$

Napier's definition of logarithms differs in several respects from the modern definition (introduced in 1728 by Leonard Euler).

In July of 1614 Napier published a small volume of 56 pages of text and 90 pages of tables entitled

Mirifici Logarithmorum Canonis Descriptio

which is best translated as *Description of the Admirable Table of Logarithm.*

In this table, calculated to 7 decimals, an introduction and the rules to be followed when using them to perform multiplication, division, or the computation of roots or powers were given. Though the word *logarithm* occurs in the title of the *Construction*, the text uses *numerus artificialis* instead. Logarithm is the name chosen by Napier from the Greek words λόγος (meaning reckoning in this context) and άριθμός (meaning number), and may be translated as 'ratio number'. It contained a statement that, if these tables were accorded the reception which Napier hoped they would get from fellow mathematicians, he would describe, in some future publication, exactly how they were discovered and the methods used to calculate them.

Napier's Descriptio was translated into English by Edward Wright and appeared in London in 1616.

Rarely in the history of science has a new idea been received more enthusiastically. Universal praise was bestowed upon its inventor, and his invention was quickly adopted by scientists all across Europe and even in faraway China, (see chapter 17). One of the first to avail himself of logarithms was the astronomer Johannes Kepler, who used them with great success in his elaborate calculations of the planetary orbits.

Napier died on 4 April 1617. His last treatise, entitled:

Mirifici Logarithmorum Canonis Constructio

was published posthumously by his son Robert in 1619. Yet it had in fact been written many years before the *Descriptio.* An English translation of the *Constructio* by William Rae Macdonald:

The Construction of the Wonderful Canon of Logarithms

was published in 1889. In the *Constructio* appears his imaginative idea of using geometry to construct a table for the improvement of arithmetic.

Figure 9.3 shows the title page of the 1619 edition of Napier's Mirifici Logarithorum Canonis Descriptio, which also contains his Constructio.

MIRIFICI

LOGARITHMO-
RVM CANONIS
DESCRIPTIO,

Ejuſque uſus, in utraque Trigonometria; vt etiam in omni Logiſtica Mathematica, ampliſsimi, facillimi, & expeditiſſimi explicatio,

ACCESSERVNT OPERA POSTHVMA;
Primò, Mirifici ipſius canonis conſtructio, & Logarithmorum ad naturales ipſorum numeros habitudines.
Secundò, Appendix de alia, eáque præſtantiore Logarithmorum ſpecie conſtruenda.
Tertiò, Propoſitiones quædam eminentiſsimæ, ad Triangula ſphærica miràſ facilitate reſolvenda.

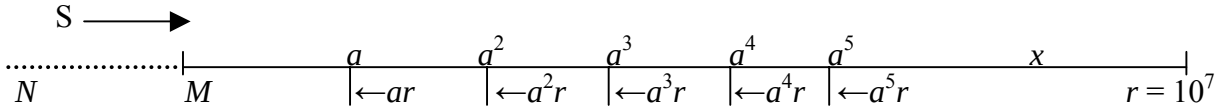
Autore ac Inventore IOANNE NEPERO, Barone Merchiſtonii, &c. Scoto.

EDINBVRGI,
EXCVDEBAT ANDREAS HART.
ANNO 1619.

Figure 9.3

9.6.3 The computation of Napier's table [9)]

His table was the outcome of about 20 years of calculations. Napier had communicated his intentions in 1594 in a letter to the Danish astronomer Tycho Brahe, who on his turn told Johannes Kepler about it.
Napier first found upper and lower bounds for log 9,999,999.
To find the lower bound, consider the point S to have moved one unit along M – r to point a, so that a – r = 9,999,999. In the same time, the point L will move more than one unit since it is moving at a constant velocity, thus M – a < O – *l*.
By definition O – *l* is the log of (M – a) 9,999,999, so that <u>1< log 9,999,999</u>
To obtain an upper bound for log 9,999,999, Napier again had a clever idea of letting the point S to start moving even before it gets to the point M. He defined a new point N, such that S takes the same time to get from N to M as it did to get from M to a.



Since the velocity of S is proportional to its distance from *r*, in the same time S will cover the same fraction of its length, no matter where it starts. Thus:

$$\frac{N-r}{N-M}=\frac{M-r}{M-a} \quad \text{or} \quad \frac{M-r}{N-M}=\frac{a-r}{M-a} \quad \text{or} \quad \mathrm{N-M}=\frac{(M-r)(M-a)}{a-r}$$

In this instance, $M-r=10^7$ while $M-a=1$, and $a-r=9{,}999{,}999$ so

$$N-M=\frac{10{,}000{,}000}{9{,}999{,}999}=1.000001$$

Because the start velocity of S at M and of L at O are equal and L has a constant velocity whereas S slows down, $N - M$ must be larger than O – l for they cover these distances in the same time, so $N - M$ > O – l or 1.000001 > log 9,999,999 or 1.000001 > log9,999,999 >1

Napier took as his value for log 9,999,999 the average of the upper and lower bounds so that log 9,999,999 = 1.0000005 a figure accurate to seven places.
He could have achieved greater accuracy by letting S move for a shorter time interval so that the upper and lower bounds N - M and M - a, would closer together.
It was now easy for Napier to obtain the logarithms of certain other numbers. He just let S move for the same time it took to get from M to a. In this time, S gets from a to a^2.
Since $M - a = 1/10^7$ part of $M - r$, $a - a^2$ is $1/10^7$ part of $a - r$.
$1/10^7(9{,}999{,}999) = .999999$
Thus, $a^2 - r = 9{,}999{,}999 - .9999999 = 9{,}999{,}998.0000001$
The point L takes the same time to go from $l - 2l$ and since L is going at constant velocity, $l - 2l$ = O – l = 1.00000005. Thus log $(a^2 - r)$ = log 9,999,998.0000001 = 2.00000010

Napier repeated the process continuous by only additions and subtractions in three tables:

In Macdonald's translation:

First table.

10000000.0000000	*Thus from radius, with seven cyphers added for greater accuracy,*
1.0000000	*namely, 10000000.0000000, subtract 1.0000000 you get*
9999999.0000000	*9999999.0000000; from this subtract.9999999, you get*
.9999999	*9999998.0000001; and proceed in this way…*
9999998.0000001	*until you create a hundred proportionals, the last of which, if you*
.9999998	*have computed rightly, will be9999900.0004950.*
9999997.0000003	
to be continued up to	Napier did not want a table with over 10,000,000 entries, so after
9999900.0004950	only 100 intervals (little steps), to obtain sufficient accuracy, he
	defined a new time interval with six cyphers added.

Second table.

10000000.000000	*A near and at the same time an easy proportion is 100000 to*
100.000000	*99999, which may be continued with sufficient exactness by adding*
9999900.000000	*six cyphers to radius and continually subtracting from each*
99.999000	*number its own 100000th part…and this table contains, besides*
9999800.001000	*radius which is the first, fifty other proportional numbers, the last*
to be continued up to	*of which, if you have not erred, you will find to be 9995001.222927*
9995001.222927	(This should be 9995001.224804).

After these 50 intervals (medium steps), follows a table of Proportionals of the Third Table, with 69 columns, the last number in the 69th column being 4998609.4034, roughly half the original number 10000000.000. Between every 69 medium time intervals (grand steps), another 21 giant time intervals (large steps), total 1449 entries and with that reached the log of 5.048.858,89.

An example of one line from his *Mirifici Logarithmorum Canonis Descriptio*

Arc/angle	sine of arc	log sine	log tan	cosine of complement	
34°40'	5688011	5642242	3687872	1954370	8224751	55°20'
	cosine of complement			log sine and sine of complement	

Napier defined Nap.log $10^7 = 0$ and Nap.log $(10^7 - 1) \approx 1$. If the value of x increases, log x decreases. To late, however, he decided that it would be easier if log 1 would be 0 in staid of 10^7 and log 10 = 10^{10} , again to avoid fractions. Than log $x.y$ = log x +log y and log x/y = log x - log y, but his weak health held him back to start new calculations. In 1615 Napier discussed this with Henry Briggs.

9.7 Henry Briggs [3)]

Henry Briggs (1561 to 1631), professor of geometry in London (Gresham College) from 1596 to 1619, then professor of astronomy at Oxford. Briggs obtained his education as St. John's College in Cambridge. In the later months of 1614 Briggs obtained a copy of Napier's *Mirifici Logarithmorum Canonis Descriptio* and exclaimed "I never saw a book which pleased me better, and make me more wonder". In March 1615 he visited Napier at his home in Scotland, and there they discussed possible modifications in the method of logarithms.
Napier at one time had proposed a table using log 1 = 0 and log 10 = 10^{10} (to avoid fractions). The two men finally concluded that the logarithm of 1 should be zero and that the logarithm of 10 should be one. Napier, who suffered badly from gout, however, no longer had the energy to put their ideas in practice. He died in 1617. To Briggs, therefore, fell the task of making up the first table of common, or Briggsian, logarithms. (In the foreword of his *Arithmetica logarithmica* of 1624 Briggs expressly attributes the important decision, log 1 = 0 to Napier). The choice of log 10 = 1, whereby 10 is determined as base and which originated with Briggs, is the most suitable for applications in the decimal system, and applicable to all the usual laws of logarithms:

log a.b = log a + log b log a/b = log a – log b $\sqrt[2]{a} = a^{1/2}$ log $\sqrt[2]{a}$ = log $a^{1/2}$ =½log a

Briggs took an arithmetic series (AS): 0 1 2 3 4 5 etc.
and a geometric series (GS) 1 10 10^2 10^3 10^4 10^5 etc.

From an AS the arithmetical average is half of the sum, e.g.: $\frac{2+3}{2} = 2{,}5$

From a GS the geometrical average is the root of the product e.g.: 10^2 and 10^3 :

$(10^2 \text{ x} 10^3) = 100\sqrt[2]{10}$ =100 x 3,1622776 = 316,22776

Briggs interposed this value in the series: 0 1 2 2,5 3 4 5
1 10 10^2 316,22776 10^3 10^4 10^5
continuing in this manner he obtained the table:

Step	numbers		AS	numbers		GS	
1	2	3	2.5	100	1000	316.22776	
2	2	2.5	2.25	100	316.22776	177.82793	
3	2	2.25	2.125	100	177.82793	133.35213	
4	2	2.125	2.0625	100	133.35213	115.47891	etc.

or written in different way with the property log $\sqrt[2]{10}$ = ½log10 = 0,5000000.
By finding $\sqrt[2]{10}$ = 3.1622776 Briggs had log 3.1622776 = log $\sqrt[2]{10}$ = ½ log10 = 0.50000
$\sqrt[4]{10}$ = 1,7782793 log1,7782793 = log $\sqrt[4]{10}$ = ¼ log10 = 0,25000
$\sqrt[8]{10}$ = 1,3335213 log1,3335213 = log $\sqrt[8]{10}$ = ⅛ log10 = 0,12500 etc.

and every number to 30 places. After 54 steps, there were 15 zero's behind the decimal point, after which Briggs could compose a table for the logarithms from 1 to 1000, inclusive, each carried out to 14 places, which he published in 1617 in *Logarithmorum chilias prima*.

In 1624 Briggs published *Arithmetica Logarithmica sive logarithmorum chiliades triginta.* This contained the common logarithms for the numbers from 1 to 20.000 and from 90.000 to 100.000, inclusive, to 14 places!

In the introduction he also explained a second method for calculating logarithms:

${}^{10}\log 100 = 2$ ${}^{10}\log 1000 = 3$ than ${}^{10}\log 561$ is between 2 and 3 or
$2 < \log 561 < 3$ or $\log 561 = 2,\ldots$
thus, for log 5612 one can suppose $3 < \log 5612 < 4$ or $\log 5612 = 3,\ldots$
So, apparently, the logarithm of any number can, with specifically conditions, be provided by counting the number of figures of the number.
Consider log 2, than, because $2^{10} = 1024$, $3 < \log 2^{10} < 4$ or $3,\ldots$ or $3 < 10\log 2 < 4$ and $0,3 < \log 2 < 0,4$ or $\log 2 = 0,3\ldots$
For more accuracy, 2 has to be calculated to a much higher power. See the table:

n	2^n	number of figures	
4	16	2	
8	256	3	
10	1024	4	
14	16384	5	to count the number of figures, it is not necessary
17	131072	6	to write out the numbers completely, only the first
20	1048576	7	5 figures will do.
. . . .	. . . . .		
1000	10715…..	302	so, till this point calculated, one can write:

$301 < \log 2^{1000} < 302$ or $301 < 1000\log 2 < 302$ or $0,301 < \log 2 < 0,302$

and for $n = 10^{10}$ to the power 10, 2^n has 301029996 figures, so log2 = 0,301029995

and than log 4 is $\log 2^2 = 2\log 2 = 0,60206$

$\log 8 = \log (2x4) = \log 2 + \log 4 = 0,90309$ (or $\log 2^3 = 3\log 2$)
$\log 40 = \log (4x10) = \log 4 + 1 = 1,60206$
$\log 400 = \log (4x10^2) = \log 4 + 2 = 2,60206$ etc.

Briggs introduced the term *mantissa*, Latin for addition, for the number behind the decimal point. The term *characteristic* for the figure before the decimal point was also suggested by Briggs, but first used in print by Adriaan Vlacq.

Briggs had the intention to calculate the deficiency values between 20.000 and 90.000. He also kindly requested "the rest of the world" to complete the table. Two Dutchmen did the job: Ezechiël de Decker and Adriaan Vlacq.

9.8 Ezechiël de Decker and Adriaan Vlacq [10)]

On October 12th 1621, **Ezechiël de Decker** of Leyden became a burgher (citizen) of Gouda. Perhaps he was born in Leiden, in 1603 or 1604, for on September 15th 1636 he states his age: about 32 years. In Gouda he was a schoolmaster till 1624 or 1625 after which he became a surveyor. Towards the end of 1629 he was living in Rotterdam where he was also occupied as a surveyor.

Adriaan Vlacq was born in Gouda roundabout 1600 and was therefore a few years older than De Decker. Adriaan received a humanistic education, and, being a mathematician, and because of his knowledge of Latin, he was able to translate Napier's *Mirifici Logarithmorum Canonis Descriptio* for De Decker. In 1631 or 1632 it was for the first time that he went to London where he established himself as a bookseller, and where he stayed till 1642. He then carried on the same trade in Paris till 1648, after which he returned to his native country. After 1650 he worked in The Hague, where he acquired his own printing-office in 1652. He died early in the year 1667.

As many of his contemporaries, Ezechiël de Decker had problems with performing calculating:

Terwijl ick inde Vermaerde Stadt Gouda Professie doende vande Meetkonst ende Rekenkonst bevondt, dat vele Leerlingen grooten schrick ende afkeericheydt hadden vande Konsten, door de groote ende verdrietighe Rekeninghen, die inde selfde voorkomen, alsoo dat de selfde by haer in verachtinghe zijn in plaets van die te beminnen, ende ooc bevondt, dat ick selfs veel tijds moste besteden int solveren van eenighe questien die my daghelijcx voorquamen, ben ick seer begheerich geweest om eenighe remedie daer toe te konnen vinden.

"While having a job of working with Geometry and Arithmetic in the famous city of Gouda, I found that many pupils were scared of and disliked the Arts, because of the large and tedious calculations to be found in these Arts. I found that they held them in contempt, instead of loving them and that I myself spent many hours solving a few problems that I had to calculate every day. I was very eager to find any solution for this".

After Vlacq translated for De Decker Napier's *Mirifici Descriptio Logarithmorum Canonis,* the two men decided to publish a complete logarithmic table and to split up this enormously work into three parts.
First in 1626, the *Nieuwe Telkonst* (New Arithmetic), taken from Brigg's *Arithmetica logarithmica,* was published. The logarithms of all integers from 1 to and including 10,000 are given, however, cut down from 14 to 10 places.

9.9 Eerste deel vande nieuwe telkonst First part of the new art of counting

Also in 1626, the *Eerste deel vande Nieuwe Telkonst* (First part of the new art of counting) by Ezechiël de Decker was published in Gouda. In the preface, dated September 4th 1626, De Decker praises Vlacq, who had translated Napier's *Mirifici Logarithmorum Canonis Descriptio* for him:
Doch also ick inde Latijnsche sprake onervaren was, badt ic den Konstlievenden Ionghman Adriaen Vlac, (…die mij met grooten yver inde Meetkonst oeffende...) dat hy 'tselfde wilde in Nederduyts overschrijven, 'twelck hy tot mijn groot contentement dede,…

"But as I was inexperienced in the Latin tongue, I begged the artloving gentleman Adriaen Vlac, (…who trained me in Geometry with great zeal) to translate the same into Dutch, which he did to my great satisfaction…"

In the *Eerste deel vande Nieuwe Telkonst* also the logarithms of all integers from 1 to and including 10,000 are given. These logarithms, transcribed from Briggs's *Arithmetica logarithmica* of 1624, are cut down to 10 decimal places, and in a supplement:
- 'Vande Telling door Roetjes' (On the counting on rods), a translation of Napier's Rabdologia;

- Appendices for this article, named: 'Veerdigh-Ghereetschap van de Menighvuldiging' (useful tools for multiplication);
- 'Van de Plaetselicke Telkonst' (on local arithmetic);
- An article he wrote on calculating interest: 'Van Coopmans Rekeninghen' (on the calculations for businessmen);
- A reprint of *De Thiende* of Simon Stevin;

In the preface De Decker reviewed a second part, with co-operation of Adriaan Vlacq, with a complete table in which the logarithms of 1 to 100.000. So, they had to calculate themselves the logarithms of 20.000 to 90.000. This had to be done fast for they wanted to be the first in the field with the new publication, because Briggs also was calculating a complete table. Vlacq put pressure upon De Decker, which leaded to disagreement.
In 1628 Vlacq published in Gouda his *Arithmetica logarihmica* with the complete table, however, without to mention the name of De Decker, with the logarithms of 1 to 100.000 to 10 places (with the same tables as in De Decker's *Tweede deel*).
With minor revisions, this work remained the basis for all subsequent logarithmic tables up to the 20th century. Not until 1924 did work on a new set of tables, accurate to twenty places, begin in England as part of the tercentenary celebration of the invention of logarithms. This work was completed in 1949.

9.10 Tweede deel vande nieuwe telkonst Second part of the new art of counting

It was only in 1920 that De Decker's promised *Tweede deel vande nieuwe telkonst* from 1627 was discovered in the library of 'La compagnie d'assurances sur la vie Utrecht', with only De Decker's name, with the logarithms from 1 to 100.000 to 10 places.
This proves that Ezechiël de Decker and not Adriaan Vlacq wrote the first complete logarithmic table.

9.11 Development of decimal notation [11)]

Jost Bürgi did not speak foreign languages, so he was not informed of developments from others and wrote in 1588 5612.299 as $561\overset{8}{2}299$ with a small circle over the unity.

The introduction of decimal fractions as a common computational practice can be dated back to the Flemish pamphlet *De Thiende*, published at Leyden in 1585, together with a French translation, *La Disme*, by Simon Stevin, (1548 - 1620), quartermaster-general of the army of the States of Netherlands. He wrote:

Dime is a kind of arithmetic, invented by the tenth progression, consisting in characters of ciphers, whereby a certain number is described and by which also all accounts which happen in human affairs are dispatched by whole numbers, without fractions or broken numbers.

Every number propounded is called COMMENCEMENT, whose sign is thus ◎.
By example, a number is propounded of five thousand six hundred twelf: we call them the 5612 *commencements*, described thus 5612 ◎, and so of all other like.
And each tenth part of the unity of the COMMENCEMENT we call the PRIME, whose sign is thus ①, and each tenth part of the unity of the prime we call the SECOND, whose sign is ②, and so of the other: each tenth part of the unity of the precedent sign, always in order one further.

As 3① 7② 5③ 9④, that is to say: 3 primes, 7 seconds, 5 thirds, 9 fourths, and so proceeding infinitely, but to speak of their value, you may note that according to this definition the said numbers are $\frac{3}{10}, \frac{7}{100}, \frac{5}{1000}, \frac{9}{10000}$.

He also proposed to modify the dividing of degrees into minutes and seconds into the decimal system.

In 1608 Pitiscus decimal point. The common use of decimal fractions can be directly traced to *De Thiende*, especially after John Napier had modified Stevin's notation into the present one with the decimal point or comma.

1) George M. Phillips
Two Millennnia of Mathematics from Archimedes to Gauss

2) John Fauval and Gray Jeremy
The History of Mathematics

3) Boyer and Merzbach
A History of Mathematics

4) Victor J. Katz
A History of Mathematics

5) Eli Maor
e The Story of a Number

6) Dr. E. Voellmy From: Kurze Mathematiker - Bibliographien
Jost Bürgi und die Logarithmen

7) Michael R. Williams
A History of Computing Technology

8) D. J. Struik
A Source Book in Mathematics, 1200 - 1800

9) Arthur Gittleman
History of Mathematcs

10) B. de Graaf From: Dutch Classics on History of Science
TWEEDE DEEL VANDE NIEVWE TEL-KONST

11) De Thiende
Simon Stevin (1585)

Chapter ten, *The Use and Application of Logarithms*

10.1 The Astrolabe and the Quadrant

Some of the ancient astronomers, like Ptolemy, undoubtedly used various sighting devices to measure the angles between stars, as the *astrolabe* (to determine the hour of the day, hypsometry – azimuth – and other astronomical and geometrical purposes), and, from these humble beginnings, the *quadrant* was slowly improved as a sighting and measurement device. The quadrant is known in many different forms with differing tables engraved upon it. One of the best known, because it has survived in the greatest numbers, is the so-called *Gunter's quadrant*, which was devised by Gunter. The scales engraved on Gunter's quadrant are the same ones found on an equally ancient instrument, the *astrolabe*, which Gunter folded into four parts and compressed so that they would fit on a quarter circle.

10.2 Galileo Galilei and the Sector

Galileo Galilei (1564 - 1642) had originally intended to take a degree on medicine, but a taste for Euclid and Archimedes led him instead to become a professor of mathematics, first from 1589 at Pisa, later at Padua, and in 1610 became first mathematician at the grand-ducal court of Tuscany. Little mathematics was included in university curricula of that time and a large proportion of the courses would now be classified as physics or astronomy or engineering applications. Galilei came close to being what we should call a mathematical practitioner. This we see in his interest in computational techniques that led him in 1597 to construct and marked a device that he called his *geometric and military compass*. In a pamphlet of 1606 with the title *Le operazioni del compasso geometrico et militaire*, he described in detail the way in which the instrument could be used to perform a variety of computations quickly without pen or paper or an abacus.
The *sector*, as the device became known, in its final form, See Figure 10.1, consisted of two arms joined together by a hinge. The arms were usually 6 to 12 in. long and were engraved with a number of different scales, most of which radiated out from the hinge pivot point. Each arm would be engraved with an identical set of scales. The usual operating mechanism for the sector was to take a measurement with a pair of dividers and open the sector until the dividers spanned the distance from a point on the scale of one arm to the identical point on the same scale of the other arm. Leaving the sector open at this angle, one can find the answer by nothing the distance between other other points on scales spanning the two arms. The sector proved so useful that it was quickly modified by the addition of several scales to make it more useful for both the land-oriented surveyor and the navigator at sea. The sector remained the main analogue calculating instrument. The use of a pair of dividers was so ingrained into scientific thinking that the full power of the device was not exploited until the middle of the 19th century. Early examples of the slide rule which still survive usually show unmistakable signs of having been used, not in the intended way, but by having had a pair of dividers pick off lengths along the logarithmic scales.

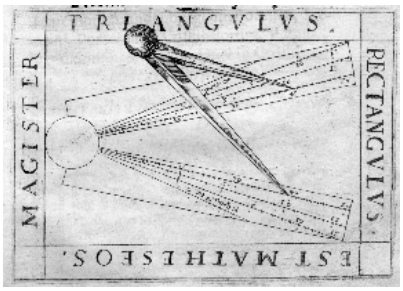


Figure 10.1

One of the driving forces behind the adoption of the sector was Edmund Gunter.

10.3 Edmund Gunter

Edmund Gunter (1581 – 1626), at the age of 18 went to Christ Church College in Oxford where he took degrees in mathematics and general arts. He was eventually selected as professor of astronomy at Gresham College at London in 1619 (at the same time that Henry Briggs was professor of geometry). About 1606 Gunter heard of the sector and, being very impressed with its use, wrote a description of it in Latin. This work was never published but was extensively copied by hand and enjoyed a wide circulation.
Gunter was primarily interested in the problems of astronomy, navigation, and the construction of sundials (the only reasonable method of telling the time in his day), all of which required large amounts of calculation involving trigonometric elements. Because of the trigonometric content of these problems, and the logarithm tables being produced by Briggs were only of marginal help, so Gunter sat down and completed the calculations for tables of the logarithms of sine's and the logarithms of tangents for each minute of the quadrant, his *Canon Triangulorum.* These eight-figure tables were published in 1620. In the *Canon Triangulorum* he also gave a description of his logarithmic *line of numbers*.

10.4 The logarithm table

The logarithm of a given number is an other number with a decimal point. The figure before this point is called the characteristic, the number after that point is called the mantissa.

The value of the characteristic is a number one less than the number of figures of the given number. Is there a decimal point in the given number itself, than only the number of figures before that point have to count, thus from 561.2 the characteristic is 2, from 5.612 it is 0. Is the number smaller than 1, e.g. 0.05612 than the characteristic is negative and in absolute value equal to the total number of zero's before and after the point, so in this case it is –2.

The mantissa has to look up in a logarithm table. Examples:

1) log 561.2 = 2.7491177 the logarithm of 561.2 = 2.7491177
2) log 5.612 = 0.7491177 the characteristic changes, the mantissa remains the same.
3) log 0.05612 = 0.7491177 - 2 or written in different way by adding 8:
   8.7491177 - 10 to simplify calculations.

If $x = (561.2)^2$ than $\log x = \log(561.2)^2 = 2\log 561.2 = 2 \times 2.7491177 = 5.4982354$
In a logarithm table we have to look up the mantissa 4982354; this belongs to 31494544. Because the characteristic is 5, the decimal point in the result belongs behind the 6th figure, in this case behind the 5 and the result is 314945.44.
If $x = (56.12)^2$ than $\log x = \log(56.12)^2 = 2\log 56.12 = 2 \times 1.7491177 = 3.4982354$.
The mantissa is again 4982354 and belongs once again to 31494544, but because the characteristic is 3 the decimal point belongs behind the 4th figure and the result is 3149.4544.
These logarithm tables were printed and used until the70s of the 20th century!

10.5 Gunter's Line of Numbers

Gunter was quite familiar with the process of using a pair of dividers to measure off various distances on the sector scales. This experience soon led him to realise that the process of adding together or subtract a pair of logarithms could be partially automated by engraving a scale of numbers, spaced in proportions to their logarithms, on a piece of wood and than using a pair of dividers to add together or subtract the two values.

Not only did this method eliminate the mental work of addition or subtraction, but it also eliminated the time-consuming process of looking up the logarithms in a table. Gunter's piece of wood soon became known as *Gunter's line of Numbers*.
Gunter's line of Numbers consisted of a simple piece of wood, about two feet long (often the shaft of a cross-staff, a simple navigation sighting instrument, invented about 1470 by Regiomontanus) see C-B on Figure 10.2, marked off with a logarithmic scale. In 1624 he published *Description and Use of the Sector, Cross-Staff and other Instruments* in which again his logarithmic scale, later known as Gunter's Line of Numbers, or Gunter's scale, or Gunter's Rule. Mostly two feet long, ca. 60 cm, piece of palm-tree wood, two in., ca. 5 cm wide. See Figure 10.3.

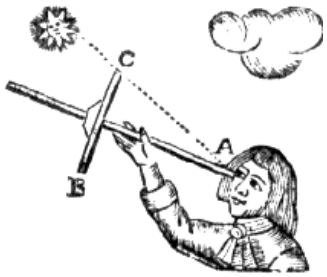
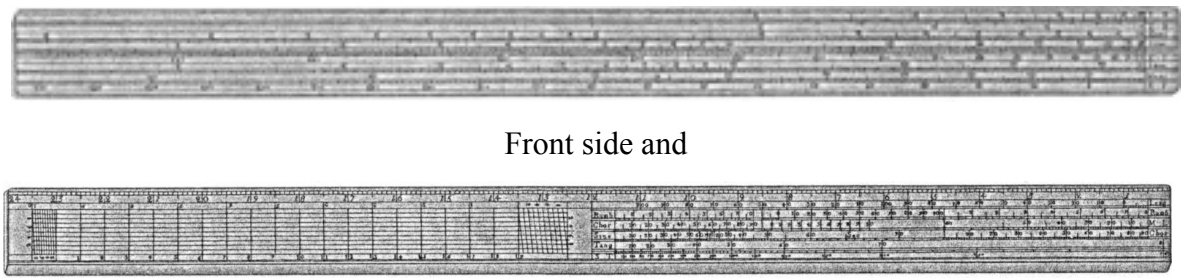


Figure 10.2



Front side and

Figure 10.3 back side of Gunter's Rule.

On the Gunter's Rule various scales are engraved, first a logarithmic scale from 1 to 100, a sine scale and a tangent scale and several scales for nautic calculation. His rule was particularly successful in lessening the computational demands of plotting a course. Captains usually were busy with a pair of dividers on sea maps, so the combination with Gunter's rule was quite natural.

Still in 1887 a textbook was written by Captain Ludwig Jerrmann: *Die Gunterskala* with the remark: "Jeder Navigateur ist im Besitz einer Gunterskala," Anyone involved with navigation possesses a Gunter's line of Numbers.

10.6 The Slide Rule and the Circle of Proportion

William Oughtred (ca. 1575 – 1660) was in 1604, after having taken a degree at Cambridge, appointed rector of a small parish, first in Surrey and a few years later in Albury, where he lived for the rest of his life. He was the bane of his bishop, being the subject of several complaints that he was a pitiful preacher because he never studied anything other than mathematics.
Oughtred offered free instruction in mathematics to anyone who was sufficiently interested to journey to his home. It must have been very good instruction, because great many of the next generation of English mathematicians received their training at Oughtred's table.
Although he had contempt for the computational side of mathematics and considered the people who used calculational instruments simply the *doers of tricks*, this did not deter him from becoming familiar with the mathematical instruments then available. He paid a visit to Henry Briggs in 1610 and, while there, met Edmund Gunter who discussed mathematical instruments with him.
Oughtred had noted that Gunter's line of numbers required a pair of dividers in order to measure off lengths of logarithmic values along the scale and quickly came up the idea that, if

he had two such scales marked along the edges of different pieces wood, he could slide them relative to each other and add or subtract two scales, thus do away with the need for a pair of dividers: the slide rule. See Figure 10.4

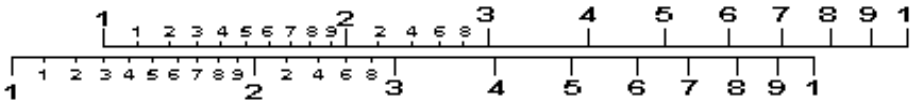


Figure 10.4 (1.3 x 2 = 2.6 or 1.3 x 3 ≈ 3.9)

He also saw that if you had two disks, one slightly smaller than the other, with a line of numbers engraved around the edge of each, they could be pinioned together at their centres and rotated relative to another to give the same effect as having Gunter's scale engraved on two bits of wood, the Circle of Proportion. Because of his general disdain for mathematical instruments he did not consider it worth his trouble, time, or effort to publish a description of how he had improved Gunter's line of numbers into a practical slide rule. He did, however, describe the system to one off his pupils, Richard Delamain, who was a teacher of mathematics.

Delamain published in 1630 a description of the Circle of Proportion in a short work *Grammelogia, or The Mathematicall Ring*, in which he described a circular slide rule he had invented. He duly patented his invention, believing that his name in history would thereby be secured.

In the same year that Delamain published his *Grammelogia,* another of Oughtred's pupils, William Foster, happened to mention that, in order to gain more accuracy when using Gunter's line of numbers, he had resorted to using a scale six feet long and a beam compass to measure off the lengths.

Oughtred then showed him how he could dispense with the beam compass by simply having two of Gunter's scales sliding over one another and also showed him a circular disc with Gunter's line of numbers marked off along the edge with two indices, like a pair of dividers, extending from the centre. The latter device, which Oughtred called his *Circles of Proportion,* he clamed to have invented sometime in 1622. Foster was so impressed that he demanded Oughtred publish a description of these inventions. Oughtred, still under the impression that these playthings were not suitable objects for the true mathematician, initially decided against it but when Delamain's book appeared claiming them as his own invention, Oughtred agreed to publish and even let Foster translate it into English so that the subject matter would be more widely distributed than if it had remained in academic Latin. Forster's book, entitled *Circles of Proportion,* came out in 1632.

10.7 The first slide rules

In 1654 Robert Bissaker already constructed a slide rule with a movable slide between two fixed parallel parts, the two outmost parts connected by cross-beams to both sides. Provided with ca. 20 scales, many of them adopted from Gunter's rule.

In 1657 Seth Partridge let construct a 'Double scale of Proportion', a two-sided slide rule, described in his book *The Description and Use of an Instrument Called the Double scale of Proportion.*

During the first centuries which followed several special slide rules were developed made of wood, brass, and even ivory especially for navigation, calculation of volumes for stowers, ship-brokers, and for commercial customs. Technicians did not use slide rules because the hand-made specimens often were crudely made.

10.8 The SOHO Slide Rule

James Watt (1736 – 1819), better known for his work on the steam engine, was responsible, at least in part, for one of the first really well-made slide rules in the very late 1700s. He had spent the early part of his life as an instrument maker at Glasgow University and was thus familiar with the techniques of engraving accurate scales upon instruments. He devised a simple slide rule consisting of one sliding piece between two fixed stocks, carefully engraved the face with four basic scales. The upper three of these logarithmic scales from 1 to 100 and the fourth from 1 to 10; the upper two for multiplication and division, the bottom two for square and root, thus the two middle scales on the slide. The scales represented in letters as: A|B A|D, or x^2 | x^2 x^2 | x in which the small lines stand for edges of the slide. See Figure 10.5.

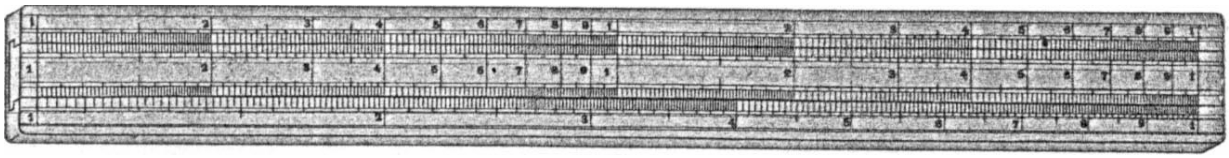


Figure 10.5 The SOHO Slide Rule

Watt let this rule be made from 1779 by a workshop in Soho near Birmingham. The rules were accurate and interesting for technicians. They were known by the name SOHO rule and now also used on the 'continent' and there frequently used for converting values in the metric system introduced by Napoleon.
Also a help in introducing the slide rule on the continent was during the period 1794 to 1799 when, due to the introduction of the metric system in France, the quadrant (of a circle) was re-divided into 100 degrees, and this required that all measures be re-computed. In 1821 Lenoir started in France the mechanical manufacture of slide rules which improved in accuracy. But the rules remained unchanged until the next improvement:

10.9 The Mannheim Slide Rule

The big breakthrough for the slide rule came in 1850 when a 19 year-old French artillery officer, Amédée Mannheim (1831-1906), designed a very simple slide rule much like that manufactured by Watt, but added the *cursor*. The cursor is a movable part, first still entire of brass, later a hair line on a piece of glass, that permit to read off values on the upper and lower scales at the same time. By that he could change the sequence of the scales: now only the upper two scales from 1 to 100, the lower two from 1 to 10, in letters, from top to bottom: A|B C|D or in formula x^2 | x^2 x | x, (the lines | indicating the slide), the basic scales, which can be found on almost all slide rules as far as the very last one. See Figure 10.6.

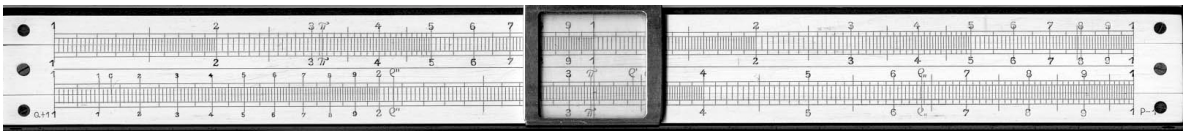


Figure 10.6 The Mannheim Slide Rule

On the bottom side of the slide are inscribed a sine scale and a tangent scale. To use these scales the slide has to turn upside down so that the sine scale or the tangent scale is opposite the A scale.

Amédée Mannheim survived his army service and was eventually appointed to the post of Professor of Mathematics at the Ecole Polytechnique in Paris, a post that did nothing to harm the ever-growing reputation of his slide rule.
From 1872 the production of slide rules started in Germany, still made of beechen or birchen. From 1888 the slide rules were made of dark mahogany, provided with a coating of white celluloid, with black lines and figures, that very improved the readability. See Figure 10.6.

The slide rule remained unpopular in North America until 1888 when several examples of the Mannheim design were imported. As late as 1901, only 50% of the engineering schools in the United States gave any attention at all to the use of the slide rule.

Two slide rules with a scale dividing for general use were available to the last, which are:

10.10 The Rietz Slide Rule

The Rietz Slide Rule, named after the German engineer Max Rietz (1872-1956), who developed this system in 1902, was in its first simple form an extending of Mannheim's slide rule with a K scale, that is a scale for cube or x^3, and a L scale with a linear dividing for the mantissa of Briggsian logarithm. Now, the sequence of the scales from top to bottom is K A | B C | D L, or in formula x^3 x^2 | x^2 x | x L, the lines | indicating the slide. See Figure 10.7

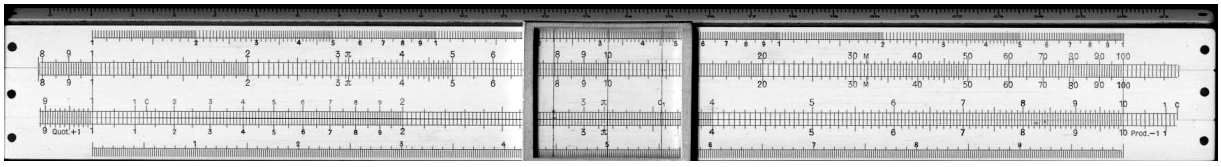


Figure 10.7 The Rietz Slide Rule

In 1925 a new scale was introduced between the B scale and C scale on the front of the slide, a C scale in reverse direction, from right to left, the CI scale or $1/x$. The new sequence of the scales now is: K A | B CI C | D L, or in formula x^3 x^2 | x^2 $1/x$ x | x L.
In 1932 a separate scale for the values of the sine and the tangent for angles smaller than $5{,}7^0$ was introduced, the ST scale, because the values for these small angles are rather equal. On the bottom side of the slide now are the S, ST and the T scales, which are to read off on the D scale by help of an index line on the bottom side of the rule. With this the development of the Rietz rule was finished and was immediately accompanied by:

10.11 The Darmstadt Slide Rule

The Darmstadt Slide Rule, was developed in 1934 by Alwin Walther (1898-1967), professor on the University of Technology at Darmstadt, Germany. To the Rietz system some scales were added and the sequence was changed. The new sequence is: L / K A | B CI C | D P | S T where L and S and T now are on the narrow sides of the rule. On the front side a new P-scale appeared, named after Pythagoras, which is $\sqrt{1-x^2}$. On the bottom side of the slide three new scales replaced the trigonometric scales: LL_1 LL_2 LL_3 . The LL scales, which are called the log log scales, are so-called exponential scales. To use these scales the slide has to turn upside down so that the LL_3 scale is opposite the D scale. With the help of these scales logarithms and powers with any given number can be calculated, for instance, $\log_e x$ and x^e.

$e = \lim$ of $(1 + \frac{1}{n})^n$ for $n \to \infty$ $e = 2.718282828\ldots$ if $y = {}^e\log x$, then $\frac{dy}{dx}$ or $y' = \frac{1}{x}$

Above the index (1) of the D (= x) scale is on the LL_3 scale e = 2,7 and above the 2 of the D scale e^2 = 7,4. In general: if a of the LL_3 scale is above the index 1 of the D scale, then above b of the D scale is a^b on the LL_3 scale.

e^x	LL_3	2.7	a^x
x	D	1	x

e^x	LL_3	a	a^b
x	D	1	b

By selecting e for the base, one can determine ln x, a function, which very frequently appears in science.

The sequences of all mentioned slide rules concerned one-sided (wooden) rules. After introducing wide two-sided rules made of synthetic material or metal, it was possible to place up to 30 different scales in various sequences.

10.12 Special Slide Rules

Special slide rules for almost every engineering speciality were common as at least one manufacturer produced a slide rule designed for its particular use. For example, a slide rule incorporating scales of atomic and molecular weights was created for chemists, see Figure 10.8 (Nestler Nr. 33). In the course of the year's special slide rules for many professions were

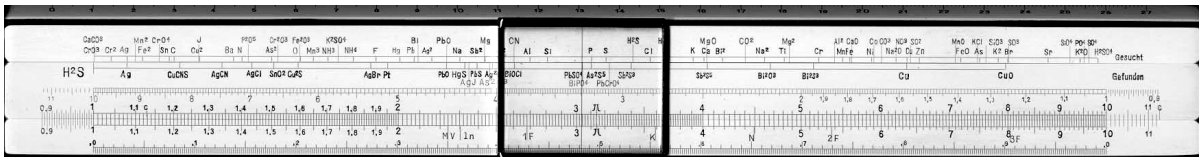


Figure 10.8 The Nestler Slide Rule for chemists

designed, for instance, air-navigation, architecture, artillery, commerce, electronics, humidity, hydraulic, local and central heating, mechanics, and surveying.

10.13 The Carbic OTIS King's Calculator

The accuracy of this slide rule was improved by modifying the basic form so that the logarithmic scales were wrapped around a cylinder in helix-form. The length of one scale is 66 in. or 168 cm., and with that, the exactness is to compare with a slide rule of 168 cm. See Figure 10.9.

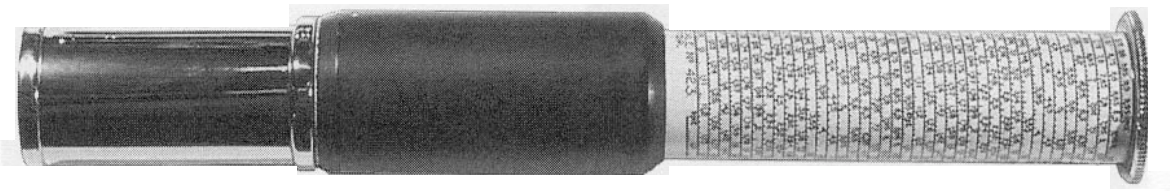


Figure 10.9 The Carbic OTIS King's Calculator

The handle is at the left, the black cylinder in middle acts as the cursor The diameter of the cylinder is 24 mm. The calculator can be pushed in to a length of 154 mm and can be slide out to 260 mm. The cylinder carries two (C and D) or three (C, D, and L) scales. The C and D scale operate for multiplication, division, etc. The L scale is graduated from 0 to 1 in 2000 equal divisions and is used in conjunction with the D scale to give direct readings of the logarithm of any number, by means of which roots and powers are obtained.

10.14 The LOGA Calculator

The LOGA calculator was available in different sizes. The diameter from 65 to 110 and 160 mm, the length from 360 to 480 and 604 mm. The calculator had only two scales, the C and D scale, both x scales. The D scale on the cylinder of the smallest one is in fact a rule of 2.4 meter long, cut into 20 pieces of each 245 mm long, overlapping each other about 50%.

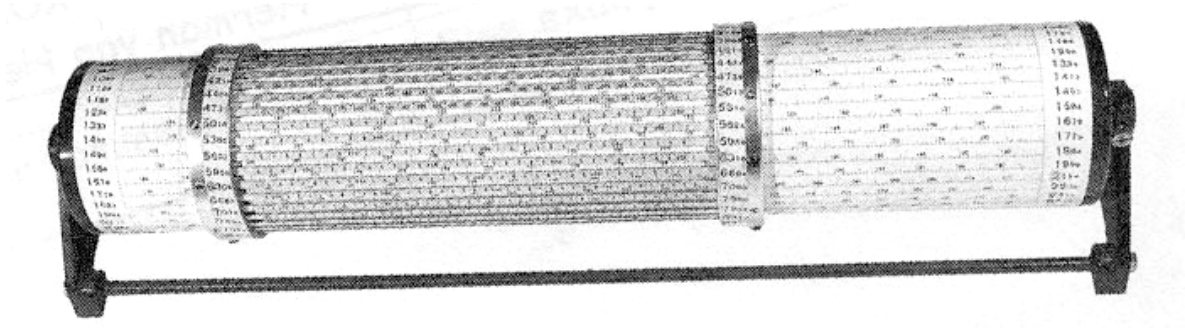


Figure 10.10 a LOGA Calculator

The C scale is 170 mm long and is mounted around the cylinder. The C scale is also cut into pieces of 200 mm long which multiplied by 40 gives the 15 meter. The length of the scale of the other LOGA's is 7.5 and 15 meter.

10.15 A watch type calculator

In figure 10.11 is represented the front side and back side of a watch type calculator.

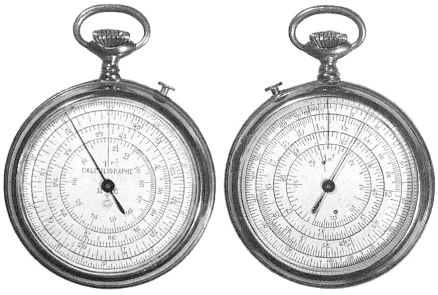


Figure 10.11

Figure 10.12

10.16 The Stanley Fuller's Spiral Slide Rule

The Fuller's Slide Rule was equivalent to a standard slide rule over 84 feet long, yet could be easily held in the hand, or operated being held by on a pin on top of its box. See Figure 10.12

1) Dieter von Jezierski
Rechenschieber eine Dokumentation Geschichte Hersteller Modelle
2) 100 Jahre Dennert & Pape ARISTO - Werke Hamburg – Altona
3) Michael R. Williams A History of Computer Technology
4) Boyer and Merzbach A History of Mathematics
5) Eli Maor The Story of Number
6) Manual of OTIS King's
7) Manual of Nestler Darmstadt Nr. 210

Chapter Eleven, *Rabdologia*

11.1 Rabdologia or calculation with Rods or Bones

John Napier frequently gave proof of his dislike to computation just like many others in the seventeenth century……as Napier himself said:

The difficulty and prolixity of calculation, the weariness of which is so apt to deter from the study of mathematics, I have always, with what little powers and little genius I possess laboured to eradicate. And with that end in view, I have published the Cannon of Logarithms

For the sake of those who prefer to work with the natural numbers as they stand, I have excogitated three other compendious modes of calculation, of which the first is by means of numbering rods and these I have called **Rabdologia**. (Rhabdos is the Greek word for a rod, and the suffix – logia means calculating)
Another, by far the most expeditious of all for multiplication, and which on that account I have not ineptly called the **Promptuary of Multiplication**, *is by means of little plates disposed in a box.* (Promtuarium is the Latin word literally means a place where things are stored ready for use).
And lastly, a third method, namely **Local Arithmetic performed upon a Chessboard**.

Napier undoubtedly used these devices in the calculation of his first table of logarithms. He did not at first consider these inventions worthy of being published; however, several friends, particularly Alexander Seton the Earl of Dumfermline and High Chancellor of Scotland, pressed him to write them down, if only to avoid others claiming them as their own.
His descriptions appeared in the year of his death (1617) in a small book entitled *Rabdologia*, in three chapters, and a number of tables to be used in geometry and mechanics. The original was written in Latin and translated into English in 1990 by William Frank Richardson.
In part one he describes and illustrates the sets of his 'numbering rods' or bones as they soon were named after the material of which the better quality bones had been made, namely bone, horn or even ivory.

11.2 Description of the bones

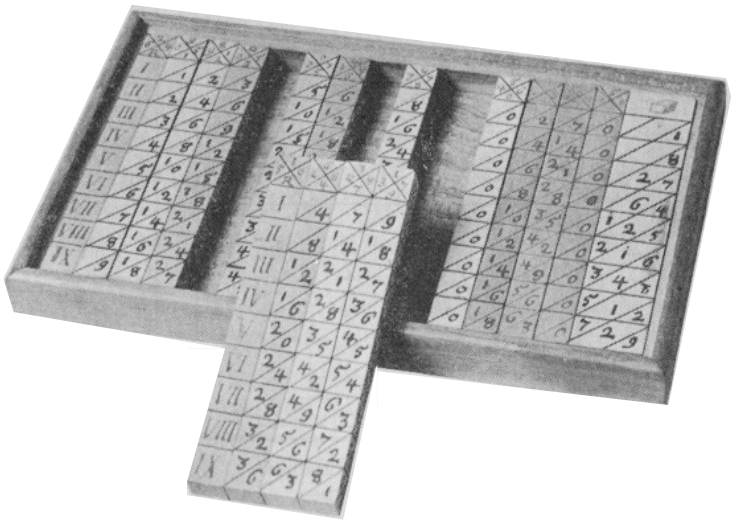


Figure 11.1 A set of Napier's bones from the Andechs monastery in Bavaria

The idea for the bones undoubtedly came from the Gelosia method of doing multiplication, as described in sector 6.10 and 8.17.5.
Instead of always drawing and inscribing the Gelosia pattern again for each calculation, Napier got the idea to take rods with a square profile and inscribe each face of them with the multiples of simple numbers.
Using 10 rods with each a clever combination of 4 tables of multiplication, he could make his calculations in an easy way.

The dimensions of the rods are in fingers: 1 finger = ¾ inch ± 19 mm;
the length of a rod is 3 fingers ~ 57 mm; the width $\frac{1}{10}$ of the length.
The sum of the units of two opposite faces always is 9, so if the table of multiplication of 2 is on top-side, 7 is on bottom-side. Of course, the same is valid for the left-hand and right-hand side.

11.2 The lay out of two numbering rods

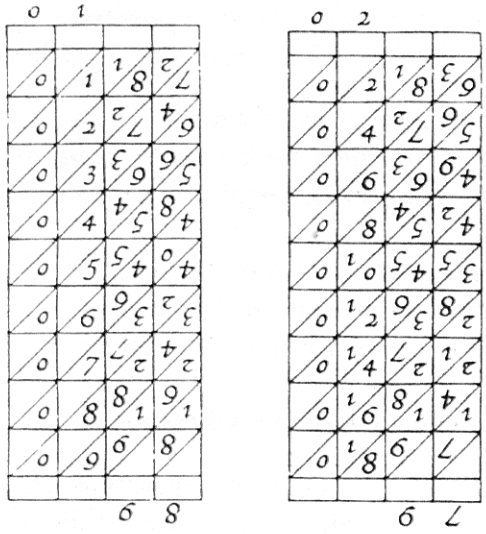


Figure 11.2
The lay out of two rods

Two of the tables on each rod are inscribed from the top downward, the two other upside down, see Figure 11.2, the pictures of the rods of 0,1,9,8 and 0,2,9,7 with their four faces unrolled and flattened out for easy reading.
The intention for this will be shown in section 11.5 when dealing with the second method of multiplication.
In the same way the other 8 rods are inscribed with the next combinations:
0,3,9,6; 0,4,9,5; 1,2,8,7; 1,3,8,6; 1,4,8,5; 2,3,7,6; 2,4,7,5 and 3,4,6,5.
The bones are not a 'calculating machine'; they only are an instrumental aid to written calculating.

For extracting square and cube roots there is an additional rod labelled *R* and 2 special plates.

11.4 Plates for extracting square and cube roots

Make from the same material as the bones a rectangular plate of the same length and thickness as the bones and as wide as three bones.
Inscribe on one face the *PRO QUADRATA* ($\sqrt[2]{\ }$) and on the other face the *PRO CUBICA* ($\sqrt[3]{\ }$) as given on the example on Figure 11.3
Make a bone marked *R* inscribed as the right-most column of the Pro quadrata plate.
The clearest way of describing their use is by following through the sample calculation as performed on 11.7 and 11.8
These illustrate the use of the bones using the notation, which would have been common in the 17th century.

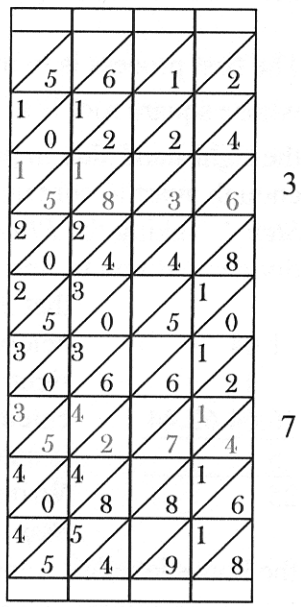
PRO QUADRATA		
1	2	1
4	4	2
9	6	3
1/6	8	4
2/5	10	5
3/6	12	6
4/9	14	7
6/4	16	8
8/1	18	9
	x^2 2x	x

PRO CUBICA		
0	1	1
0	8	4
2	7	9
6	4	16
1/2	5	25
2/1	6	36
3/4	3	49
5/1	2	64
7/2	9	81
	x^3	x^2 x

Figure 11.3

The use of Napier's bones spread rapidly and, within a few years, examples could be found in use from Europe to China, see chapter 17.

11.5 Multiplication with the bones



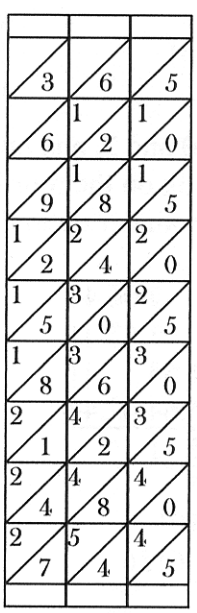
To perform the multiplication of 5612 by 37 one would simply select the bones headed 5, 6, 1, and 2 and place them side by side. Than read the partial products of 5612 times 7 by adding up the digits in each parallelogram to obtain each digit of the partial product (see 7th row), from right to left: 4; 1 + 7; 2; 4 + 5 and 3, written as 39,284. Than read the partial products of 5612 times 3(0), (see 3rd row: 6; 3; 8; 1 + 5 and 1 = 16,836(0). Than add together the partial products:

$$\begin{array}{rr} 7 \times 5612 = & 39284 \\ 3(0) \times 5612 = & \underline{16836} \\ \text{add:} & 207644 \end{array}$$

Alternatively and as a test of the preceding multiplication, turn over the entire table of rods *en bloc*, and you will find at the head of the table the number 4387, the opposite of the original number (5612). Now write down diagonally 7 times and 3 times this number, that is 30,709 and 13,161 with the lesser of the multiplicands, 37,directly underneath. Add all these up and they will give 162,356. Subtract this from 370,000 (that is, from the same multiplicand increased by as many ciphers as there are numbers in the other multiplicand), and the remainder is 207,644, the same number as above.

$$\begin{array}{r} 30709 \\ 13161 \\ + \quad 37 \\ \hline 162356 \\ \underline{370000} \\ 207644 \end{array}$$

11.6 Division with the bones



Write down on paper the dividend (the number to be divided) 589,475 and for the divisor 365 select the bones headed 3, 6, and 5 and place them side by side. Among all the multiples of 365 the simple 365 itself is the closest to but less than 589 (≤ the first figures of the dividend). The number 356 should therefore be written under the 589 and subtracted from it. The remainder 224 is to be written above the 589, and the quotumus (or index) of the simple, namely unity, is to be placed in the quotient. This upside-down form of subtraction (*galley method*, see § 7.10) was one of the common methods in use in Napier's day.

```
      182
       54
      224
365 | 589.475 | 1615
      356
      2190
        365
        1825
```

Step 2. In the sixth interval of the set of bones you will find 2,190, which is six times the divisor and ≤ 2,244, which appears above the dividend. The number 2,190 should therefore be written under the 2,244 and subtracted from it. This gives a remainder of 54 which is to be written above, while the index of this multiple 6, is to be added to the quotient.

Step 3. This repeats step 2. Look for the multiple ≤ 547. This is once again the simple itself, 365. This is subtracted from 547 to give a remainder of 182, which is written above, and the index of the simple, which is the unity, is to be placed in the quotient.

Step 4. Lastly, the multiple closest to the number 1,825 is to be sought, and a figure equal to this, 1,825, will be found in the fifth interval. This number should be written beneath the other and subtracted from it, which leaves a remainder of nothing. The figure 5 should be added to the quotient. Thus the desired quotient is 1,615.

11.7 Extracting square roots with the bones

PRO QUADRATA		
0/1	2	1
0/4	4	2
0/9	6	3
1/6	8	4
2/5	10	5
3/6	12	6
4/9	14	7
6/4	16	8
8/1	18	9

The first process is to mark off with dots the digits of the given number whose square root is to be extracted, 26,646,244 ($\sqrt[2]{26646244}$), from the right-hand side in pairs, and underneath these draw two lines far enough apart to contain the root.

Step 1. Taking the *PRO QUADRATA* plate, one looks in the diagonally divided column to find the largest number that is ≤ the first pair of digits that is 26; in this case it is 25, which is then entered below the horizontal lines, and the digit from the right-most column, 5, the first digit of the root, is entered between the lines. The difference 26 – 25 = 1 is entered above the original number (26). Write the next two digits (64) beside that 1→164.

```
 1 64
   ||
26.64.62.44
 5
25
```

Step 2. Double this root, which gives 2 x 5 = 10, find the bones labelled 1 and 0, and lay the *Pro quadrata* plate beside them. Look on the three left-most columns for the multiple ≤ the remainder up to the 2nd dot, 164. You will find this to be 101, which is 1 times and 1 is the second digit of the root and is entered between the lines, 101 is entered below the horizontal lines. The difference 164 – 101 = 63 is entered above the number164. Write the next two digits (62) beside that 63 → 6362

		PRO QUADRATA		
0/1	0/0	0/1	2	1
0/2	0/0	0/4	4	2
0/3	0/0	0/9	6	3

```
   63 62
 1 64 ||
26.64.62.44
 5  1
25
  101
```

Step 3. Double the two digits of the root 51 which gives 102, find the bones labelled 1, 0 and 2, and lay the *Pro quadrata* plate beside them. Look on the four left-most columns for the multiple ≤ the remainder up to the 3rd dot, 6362. You will find this to be 6156, which is 6 times and 6 is the 3rd digit of the root and is entered between the lines, 6156 is entered below the horizontal lines. The difference 6362 – 6156 = 206 is entered above the number 6362. Write the next two digits (44) beside that 206 → 20644

			PRO QUADRATA		
0/1	0/0	0/2	0/1	2	1
0/2	0/0	0/4	0/4	4	2
0/3	0/0	0/6	0/9	6	3
0/4	0/0	0/8	1/6	8	4
0/5	0/0	1/0	2/5	10	5
0/6	0/0	1/2	3/6	12	6
0/7	0/0	1/4	4/9	14	7

```
     2 06 44
    63 62 ||
  1 64    ||
 26.64.62.44
   5  1  6
 25
   101
    6156
```

Step 4. Double the three digits of the root 516 which gives, 1032 find the bones labelled 1, 0, 3 and2, and lay the *Pro quadrata* plate beside them.

Look on the five left-most columns for the multiple ≤ the remainder up to be 20644, which is 2 times and 2 is the 4th digit of the root and is entered between the lines, 20644 is entered below the horizontal lines. The difference = 0. Thus the required root is 5162

				PRO QUADRATA		
0/1	0/0	0/3	0/2	0/1	2	1
0/2	0/0	0/6	0/4	0/4	4	2
0/3	0/0	0/9	0/6	0/9	6	3

11.8 Extracting cube roots with the bones

The first process is to mark off with dots the digits of the given number whose cube root is to be extracted, 122,763,473 , ($\sqrt[3]{122763473}$) from the right-hand side in groups of three, and underneath these draw two lines far enough apart to contain the root.

Step 1. Taking the *Pro cubica* plate and lay the *R*-bone beside it. Look in the two left-most columns of the *Pro cubica* plate for the multiple (3rd power) ≤ the first three digits of the given number, that is 122; in this case it is 64, which is entered below the lines, and the digit from the *R* bone, 4, the first digit of the root, is entered between the lines. The difference 122 – 64 = 58 is entered above the original number (122). Write the next three digits 763 beside the 58 → 58763.

```
 58 763
122.763.473
  4
 ---------
 64
```

PRO CUBICA		*R*
0/0	1	1	1
0/0	8	4	2
0/2	7	9	3
0/6	4	16	4
1/2	5	25	5
2/1	6	36	6
3/4	3	49	7
5/1	2	64	8
7/2	9	81	9

Step 2. Three times the root = 3 x 4 = 12. Select the bones labelled 1 and 2 and place them on the right-hand side of the *Pro cubica* plate. Three times the square of the root = 3 x 4^2 = 48. Select the bones labelled 4 and 8 and place them on the left-hand side of the *Pro cubica* plate. Look in the four left-most columns for the number ≤ 58763 = 43929 and 9 is the second digit of the root; and 9^2 = 81 or 8(0) + 1; read off on the 2 right-most bones 12(0) on line 1and 96(00) on line 8. Now add 43929, 120 and 9600 = 53649. is entered below the lines. The difference 58763 – 53649 = 5114 is entered above the number 58763. Write the next three digits 473 beside the 5114 → 5114473.

```
  5 114 473
 58 763 |||
122.763.473
  4   9
 ---------
 64
 53 649
```

```
  43929
    120
+ 9600
  53649
```

			RO CUBICA		*R*			
0/4	0/8	0/0	1	1	1	0/1	0/2	120
0/8	1/6	0/0	8	4	2	0/2	0/4	
1/2	2/4	0/2	7	9	3	0/3	0/6	
1/6	3/2	0/6	4	16	4	0/4	0/8	
2/0	4/0	1/2	5	25	5	0/5	1/0	
2/4	4/8	2/1	6	36	6	0/6	1/2	
2/8	5/6	3/4	3	49	7	0/7	1/4	
3/2	6/4	5/1	2	64	8	0/8	1/6	9600
3/6	7/2	7/2	9	81	9	0/9	1/8	9^2 = 81 8(0)+1

Step 3. Three times the first 2 digits of the root = 3 x 49 = 147. Select the bones 1, 4, and 7 and place them on the right-hand side of the *Pro cubica* plate. Three times the square of the first 2 digits of the root = 3 x 49^2 = 7203. Select the bones 7, 2, 0, and 3 and place them on the left-hand side of the *Pro cubica* plate. Look in the 6 left-most columns for the multiple ≤ 5114473 = 5042443 and 7 is the 3rd digit of the root. 7^2 = 49 or 40 + 9; 4 gives 588(00) and 9 gives 1323(0). Add now 5042443 +58800 and 13230 = 5114473: the difference = 0. Thus the required root is 497.

				PRO CUBICA			*R*				
0/7	0/2	0/0	0/3	0/0	1	1	1	0/1	0/4	0/7	
1/4	0/4	0/0	0/6	0/0	8	4	2	0/2	0/8	1/4	
2/1	0/6	0/0	0/9	0/2	7	9	3	0/3	1/2	2/1	
2/8	0/8	0/0	1/2	0/6	4	16	4	0/4	1/6	2/8	58800
3/5	1/0	0/0	1/5	1/2	5	25	5	0/5	2/0	3/5	
4/2	1/2	0/0	1/8	2/1	6	36	6	0/6	2/4	4/2	
4/9	1/4	0/0	2/1	3/4	3	49	7	0/7	2/8	4/9	7^2 = 49 4(0)+9
5/6	1/6	0/0	2/4	5/1	2	64	8	0/8	3/2	5/6	
6/3	1/8	0/0	2/7	7/2	9	81	9	0/9	3/6	6/3	13230

In an appendix to his work *Rabdologia* Napier gives two other methods of calculation, *The High-Speed Promptuary for Multiplication* and *Location Arithmetic as performed on a Chessboard*. These will be described in the following sections.

11.9 The High-Speed Promptuary for Multiplication

For this system you need two different kinds of strips, ‘vertical’ and ‘horizontal’ strips. These strips should be one finger wide (~ 19 mm) and eleven fingers long (~209 mm). The strips should be made of a solid material. (Napier suggests ivory!)

1) The vertical strips, or ‘digit’ strips should be one quarter of a finger thick (~5mm). At the top end a margin of ⅔ of a finger, at the other a lesser margin of ⅓ of a finger, in the middle 10 squares. In the larger margin one of the digits from 0 to 9 should be written. Napier suggests at least ten copies of each species of strip for each digit from 0 to 9!

The design of each square is according a fixed pattern: a diagonal line from the lower left to the upper right, thus performing two triangles. Then divide each square into 9 smaller squares and draw a diagonal line in each small square. Each large triangle now contains nine places; these should be marked with the first nine letters of the alphabet in the order indicated by Figure 11.3

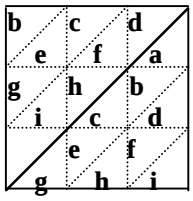
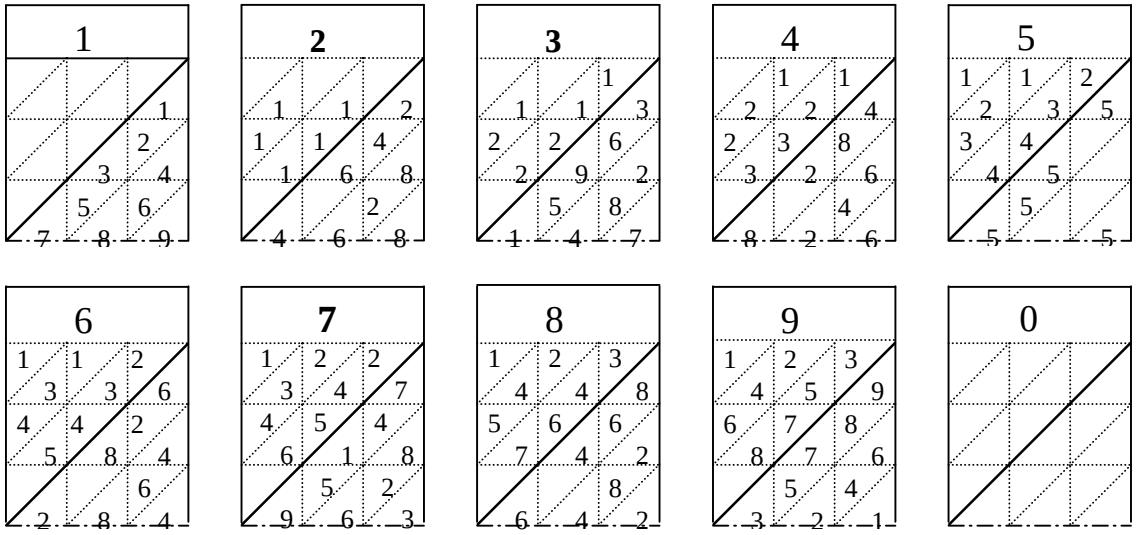


Figure 11.3

As the unities of each table of multiplication on the rods are inscribed from the top downward in the triangle right-below, the same unities should be inscribed on each square of the strips in each small triangle from a to i and the tens from b to i in such a way that there is together one product of the table in the small triangles with the same letter:



If any multiple has no left-hand figure or if any right-hand figure is a cipher, leave that place vacant. For the correctly inscribed strip for number 4 see Figure 11.4.

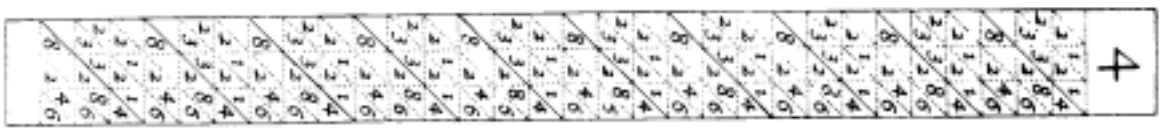


Figure 11.4

2) The horizontal strips, or 'window' strips should be as long and wide as the vertical strips but ⅛ of a finger thick (~2.4 mm) These perforated strips serve to block out the multiples on the vertical strips that you do not need and to reveal through windows or holes the ones that you do. To determine the places of the required triangular perforations you have to 'project' Figure 11.3 on the strips. For the strips labelled 1, the places of the windows correspond with the place of the letter a; for the strips labelled 2 you need b; for 3 it is c etc. No cuts at all are made in the strips labelled with a cipher (in the larger margin).

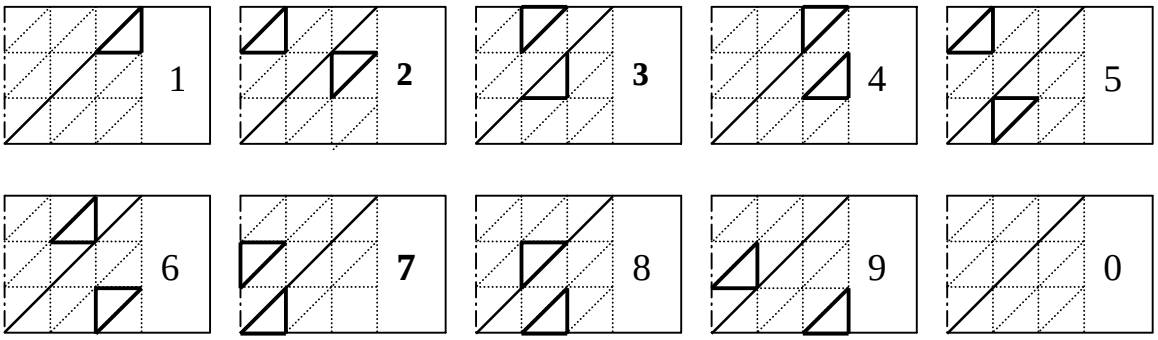


Figure 11.5

For the right-most square of the 10 horizontal strips, see Figure 11.5 For the correctly perforated number 7 strip, see Figure 11.6.

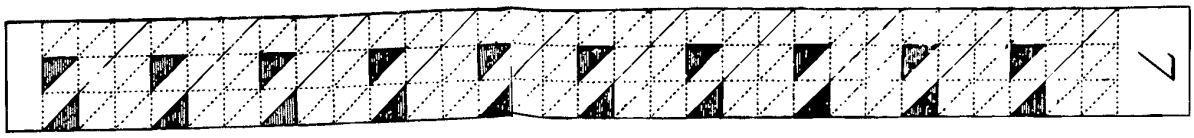


Figure 11.6

11.10 The Promptuarium box

To store the strips you need a box or *Promptuarium* with slots.
The ten vertical strips in the slots next to each other, the ten horizontal strips next to each other at right angles and below them.
To perform a multiplication with large numbers you need many strips.
In his *Rabdologia* Napier gives as an example a box with 10 specimen of each strip. Such a complete set would comprise 200 strips, representing a small fortune. See Figure 11.7

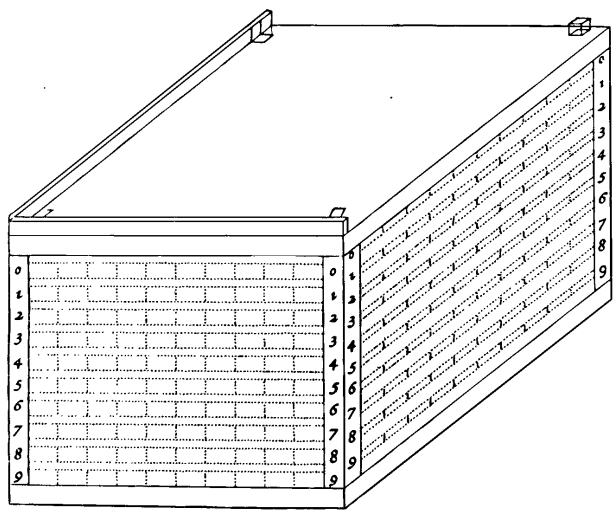
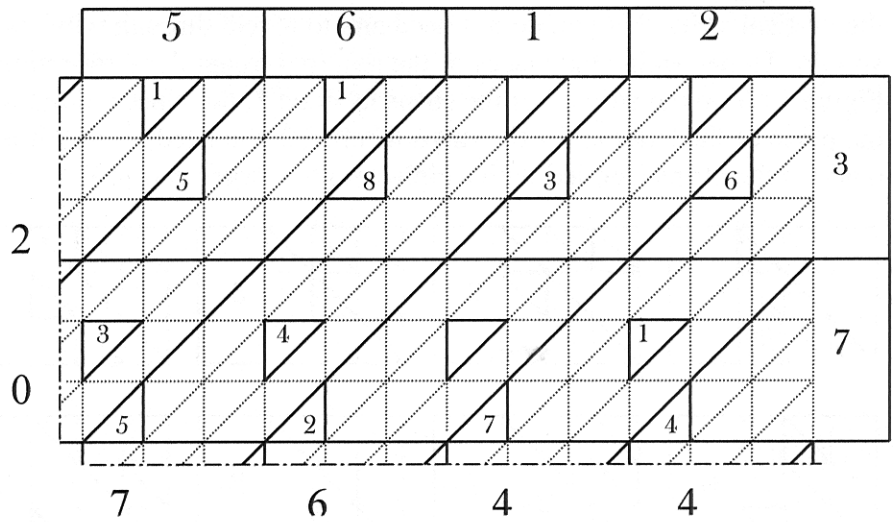


Figure 11.7

11.11 Multiplication on The High-speed Promptuary for Multiplication

Example of a multiplication : 5612 x 37
The product can be read by adding up the figures showing through the hole in each of the main diagonals.
From right to left:
4; 6+1+7=14;
4 'write down' etc.
thus the product is 207644

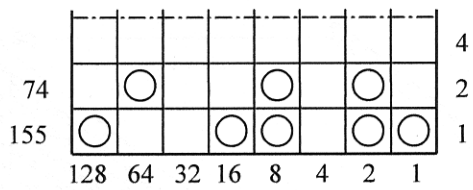


According Napier, division with the Promptuary for Multiplication is possible by multiplication with the opposite: 20 divided by 8 = 20 x $\frac{1}{8}$. To find these opposites Napier gives tables. Finaly Napier describes his Location Arithmetic as performed on a Chessboard.

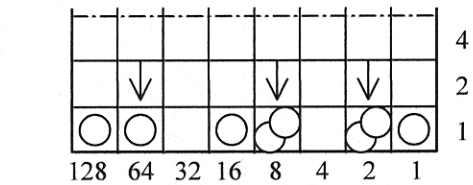
11.12 Location Arithmetic as performed on a Chessboard

This is a method of calculating by moving counters across a chessboard of arbitrary size, with columns and rows labelled by the doubling series 1, 2, 4, 8, 16, 32 ...(successive powers of 2). Any positive integer is uniquely represented as a sum of the powers of 2. Each number is expressed by placing counters on a row of the board.

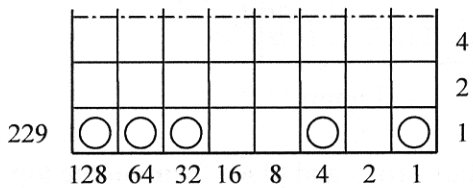
11.12.1 Addition on the board



addition 74 + 155, step 1



addition 74 + 155, step 2

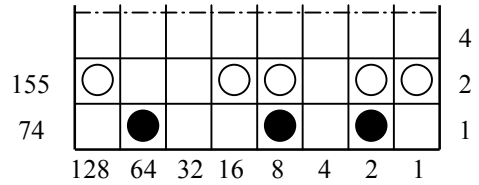


addition 74 + 155, step 3

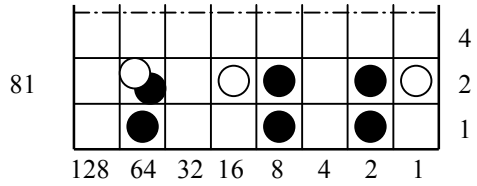
If we want to add 74 + 155, than the first row shows 155 as the sum of 128 + 16 + 8 + 2 + 1 (1 on the right-bottom cell), the second row shows 74 as the sum of 64 + 8 + 2 (see step 1).
If you think of each counter as 1 and the empty cells as 0, then 155 is represented in binary notation as location numerals according to Napier, as 10011011. It is the world's first binary 'computer', and it came almost 100 years before Leibniz explained how to calculate with binary numbers.
To add the 2 numbers, first move all the counters down to the bottom row (see step 2).
Then clear the row of multiple counters on a cell by the following procedure. Start at right, taking each cell in turn, remove every pair of counters on a cell and replace them with a single counter on the adjacent cell to the left. We shall call this 'halving up'. The final result after clearing is the binary number 11100101, or 229 in decimal notation (see step 3).

11.12.2 Subtraction on the board

Subtraction is almost as simple. Represent the minuend (155 ○) on the second row and the subtrahend (47 ●) on the bottom row, (see step 1). If there is an empty cell on the second



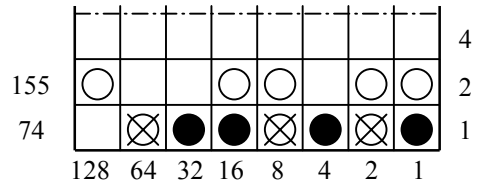
subtraction 155 - 74, step 1



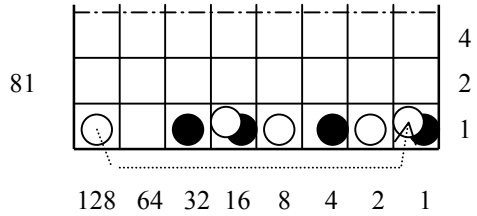
subtraction 155 - 74, step 2

row above a counter on the bottom row, than 'doubling down' on the second row: removing a counter and replacing it with two counters on the next cell to the right. (See step 2) Now there is above each counter on the bottom cell a counter on the second cell: ● remove them and the second row shows in binary notation the difference 1 0 1 0 0 0 1 or 81 (○)

A different subtraction is to complement the subtrahend. A number is complemented by placing a counter on each of its empty cells (see ●) and removing all counters originally there



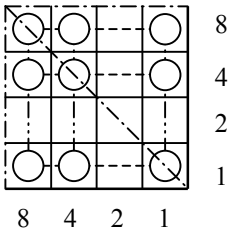
subtraction 155 - 74, step 1



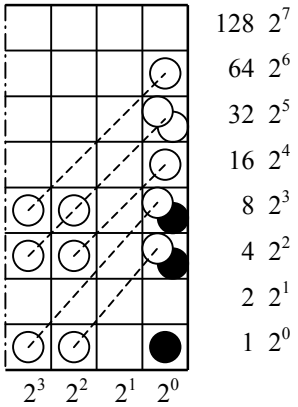
subtraction 155 - 74, step 2

(See step 1⊗). Add the two numbers, clear the row by halving up, and then transfer the counter at the extreme left to the extreme right. Clear again if necessary (see step 2).

11.12.3 Multiplication on the board



. step 1



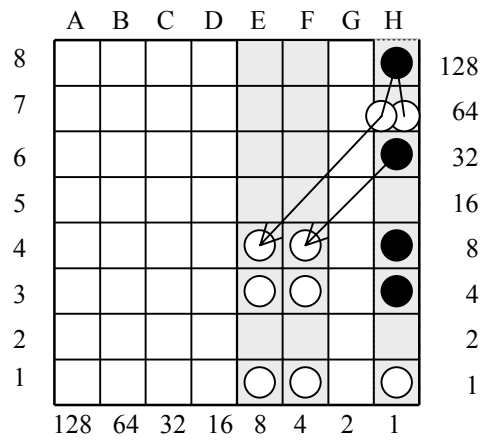
Step 2

Multiplication is delightfully easy. In this example we use a square: 13 x 13. Note: as the example is a square, there is a diagonal of symmetry, see –·–·–.
One number 1101 is indicated by marking the proper columns –··–··, the other number by marking the proper rows ------ . A counter goes on each cell at the intersection of a marked row and column, (see step 1).
Than every counter not on the column at the extreme right is moved diagonally up and right (like a chess bishop) until it is on the rightmost column. The result is shown (step 2).
To express the desired product in binary notation, clear the column by halving up. The result is 10101001 or 169.
Counters on the first row keep their values when moved to the right (2^0), counters on the second row double in value (2^1), counters on the third row quadruple in value (2^2) and so on. The procedure is equivalent to cross multiplication by adding the exponents:

$$
\begin{array}{lllllll}
13 \times 13 = & 2^3 & + & 2^2 + 2^0 & & & \\
 & 2^3 & + & 2^2 + 2^0 & & & \\
\hline
 & 2^6 & + & 2^5 & + & 2^3 & \\
 & & & 2^5 & + 2^4 & + & 2^2 \\
 & & & & 2^3 & + 2^2 & + 2^0 \\
\hline
 & 2^6 + 2(2^5) + 2^4 + 2(2^3) + 2(2^2) + 2^0 & & & & &
\end{array}
$$

Operation of cross multiplication by adding exponents.

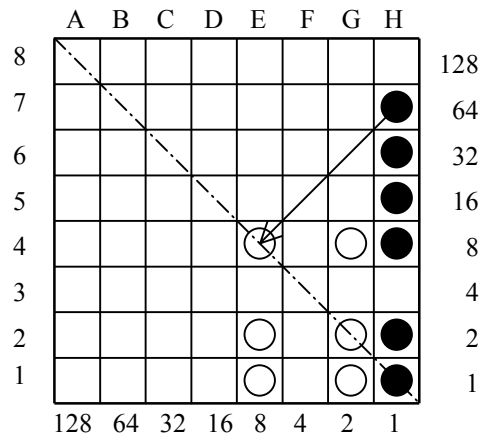
11.12.4 Division on the board



The procedure of division (as one would expect) is the reverse of multiplication. Example 172 ÷ 13. The divisor, 13 (1101), is marked by coloured the proper columns. The dividend is indicated by counters on the column at the extreme right, (H): 172 = 128 + 32 + 8 + 4 (see ●). You must now move the dividend counters like a chess bishop, down and left, to produce a pattern that has counters (one to a cell) only on marked columns, and each marked column must have its counters on the same row. Start with the counter on H8 and move it diagonally to the leftmost marked column: E5. You see that you cannot proceed to form the disired pattern on E5, F5 and H5.

Return the counter to its original cell, double down (⇓) and one counter from H7 to E4 and the counter from H6 to F4: the first row is complete. Continue in this manner: H7(⇓) 2xH6; $H6_1 \rightarrow E3$; $H6_2 \rightarrow 2xH5$ and $H5_1 \rightarrow F3$: second row is complete. $H5_2$(⇓) 2xH4; $H4_1 \rightarrow E1$; $H4_2$ (⇓) 2xH3; $H3_1 \rightarrow F1$; $H3_2 \rightarrow 2xH2$; 2x H2(⇓) 4xH1: third row complete. The rows containing counters are marked on the right margin, symbolising 1011 or 13, the correct quotient. The three extra counters, left over, represent the remainder, or give the fraction 3/13.

11.12.5 The extraction of square roots



As in division, counters on the rightmost column represent the number: $\sqrt[2]{123}$

123 = 64 + 32 + 16 + 8 + 2 + 1, see ●. We must move counters diagonally down to produce a pattern with two properties:(1) every column with counters must have its counters on the same rows, and (2), the pattern must have bilateral symmetry along the diagonal passing through the board's lower tight corner, see −·−·. This ensures, of course, that multiplier and multiplicand are identical. Start with the top counter and see if you can move it to the diagonal of symmetry: H7→E4. The counter on H6→F4: there is no counter for E3: so H6(⇓)2xH5; $H5_1 \rightarrow E_2$; $H5_2 \rightarrow G4$; $H5_3$(⇓)2H4; $H4_1 \rightarrow E1$; $H4_2$(⇓)H3; $H3_1 \rightarrow G2$; $H3_2$(⇓)2H2; $H2_1 \rightarrow G1$; $H2_2$(⇓)2H1. The rows and columns containing counters are marked on both margins, symbolising 1011 or 8 + 2 + 1 = 11, the root of 123. You will have 2 two counters left over, they represent the remainder.

RABDOLOGY, by John Napier
Volume 15 in the Charles Babbage Institute Reprint Series for the History of Computing

Chapter twelve, *Early versions on Napier's bones*

12.1 The use of cylindrical bones

Users of a standard set of Napier's arithmetic bones were aware of the physical problems involved – such as locating the correct bones and having some convenient device to ensure correct positioning.

Various inventors had the idea to take a series of cylinders, which have the values of the complete multiplication table (including zero) imprinted around their circumference. Several of these cylinders were then housed in a box that would permit turning the cylinders and bring the proper columns in view. See Figure 12.1
Independent of each other versions with cylindrical bones were invented by:

Figure 12.1

Gaspard Schott (1608 - 1666), a German Jesuit. He could examine individual cylinders through slits cut in the top of his box. The box should also have a lid with tables for addition and subtraction on the inside to aid the operator. He added to the box a number of other tables for geometry, surveying, calendrical calculation, and planetary motions.

Pierre Petit, a French mathematician, intendant of fortifications, and friend of Pascal.

René Grillet, a watchmaker in Paris, recorded in his *Curiositez mathematiques* his own inventions, including a new arithmetical machine. Grillet recognised a serious difficulty in the use of the Rhabdology: "it often takes as much time to choose and arrange the rods as would be required to do the problem on paper." He found Petit's drum was "large and extremely clumsy." Grillet claimed that his own *machine de bois* – offered advantages in portability and price. Its design and ease of operation not only reduced multiplication and division to addition and subtraction but also simplified the extraction of roots.

Sir Charles Coterell in 1667 constructed a box, the lid of which consisted of a tiny wire-and-bead abacus to be used when adding together the partial products. The set came with a small brass wand to move the abacus beads and a brass 'ladder' to act as a cursor for the bones. Other users covered the space between the cylinders with a strip as indicated in figure 12.1.

Although it was an interesting attempt at making the bones easier to use, the system proved to be a failure. The parallelograms containing the digits to be added together span two adjacent bones, and the space required to mount the cylinders meant that these digits were widely separated. This led to a greater tendency to make mistakes and the device was soon abandoned. But no invention was so ingenious as Schickard's calculator.

12.2 Wilhelm Schickard

Wilhelm Schickard (1592 – 1635), was Professor of Hebrew, Oriental Languages, Mathematics, Astronomy and Geography, also a Protestant minister in Tübingen, a good painter and a good enough mechanic to construct his own astronomical instruments and an engraver skilled enough to provide some of the copper plates used to illustrate Kepler's *Harmonices Mundi* (1619).

Kepler met Schickard at a visit to his mother in Tübingen (where Kepler's mother was accused of being a witch and thrown into jail), in the winter of 1617 and showed Schickard his set of Napier's bones and a logarithmic table.
This seems to have inspired Schickard to consider the design of a 'machine' which would incorporate both a set of Napier's bones and a mechanism to add up the partial products in order to completely automate the multiplication process.

12.3 Schickard's calculator

Schickard's calculator is composed of the following three units (top to bottom):

- the multiplication unit, (that also enables to perform division by multiplying with the reciprocated value of the divisor),
- the adding machine (the adder) with six horizontal dials and
- the memory unit with six knobs, see Figure 12.2

These three units are independent of one another; they are not mechanically inter-connected.
The multiplication unit consists of six vertically positioned, turnable cylinders, which have the values of the complete multiplication table (including zero) imprinted around their circumference.

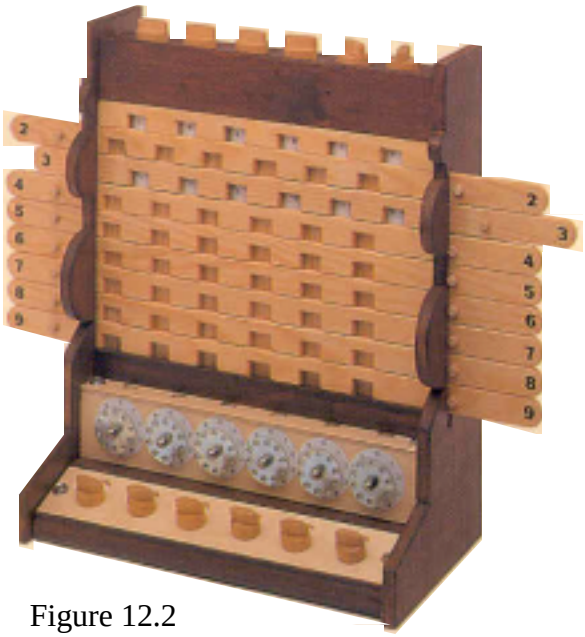


Figure 12.2

Let's look at performing a multiplication e.g. 5612 x 37.
First turn the knobs on top to enter the value of the multiplicand. This number will subsequently appear in the permanently open windows of the fixed upper slide. In the rest position, the calculator's window slides are pulled fully toward the right. Push the window slide that bears the number of the rightmost multiplier digit (7) to the left. The slide windows should show 0| 0 0| 0 3| 5 4| 2 0| 7 1| 4.
At the first adder dial (counting from right to left) dial the digit 4 to enter the units position.
At the second adder dial, dial the digit 1 (one ten); in the same position, dial the digit 7 (seven tens); notice that the window above will now show the result 8 (1 + 7).
At the third adder dial only the digit 2 is to dial to enter the hundreds position, and so on.
Finally, the adder's windows should display 39284.
At the memory unit of the calculator, dial the number 7 into the units position (right-most memory dial) as a reminder that you have just worked off the multiplier's units position, which was the digit 7, then restore the slide number 7 by pulling it all the way to the right.
Now multiply with 30 (multiplier digit 3) by pushing slide number 3 all the way to the left.
The windows should show 0| 0 0| 0 1| 5 1| 8 0| 3 0| 6. These partial products must be added to the already existing amounts, again from right to left, but this time beginning in its ten position because the unit's position of the multiplier has already been processed, which is indicated by 7 in its position. Proceed as follows:
At the second adder dial (counting from right to left) dial first 6, then 3 in the next position.
At the fourth adder dial, enter 8, at the fifth 6, and at the sixth 1. This completes the addition of the partial products.

At the memory unit, dial 3 into the tens position as a reminder for the second multiplier digit that you have just finished processing. Then, pull slide number 3 all the way to the right.

The adder lets the operator perform subtractions simply by turning the dials in the reverse direction. The calculator automatically takes any negative carries into consideration.
At performing a division the dividend have to set in the adder, the divisor with the cylinders and the quotient in the 'memory'.

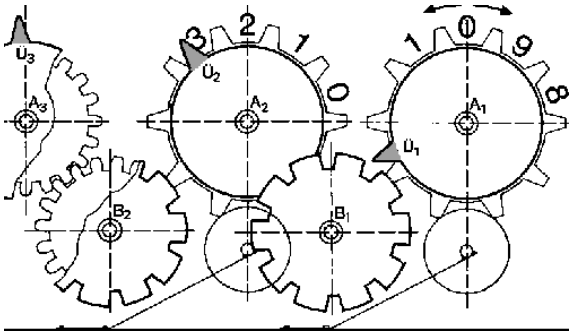


Figure 12.3

Figure 12.3 is a part of the mechanism of the adder. Wheel A_1 is turned by hand. The big teeth only serve to hold the wheel in the correct position, with help of the spring and roller. The dark tooth D_1 makes the carry in co-operation with wheel B_1. Next to B_1 is a wheel with 20 tooth that with a similar wheel next to A_2 completes the carry.

In two letters to Kepler, of September 20, 1623 and February 25, 1624, Schickard described his machine, supplied with several drawings showing its construction. He told Kepler that a second machine, which was being made for him, had been accidentally destroyed when a fire levelled the house of a workman Schickard had hired to do the final construction.
None of the 2 machines has survived. During a conference about the history of mathematics – which was conducted in April 1957 at the Institute for Mathematical Research at Oberwolfach (Black Forrest region, Germany) – Franz Hammer drew the audience's attention to drawings by Wilhelm Schickard depicting his calculating machine. Dr. Bruno Baron von Freytag Löringhoff, a professor for philosophy at the University of Tübingen and conference attendee, succeeded in unravelling the internal mechanism of his calculator from these drawings while the conference was still in progress. Later he prepared a reconstruction, See Figure 12.1

12.4 Samuel Morland [1)]

Samuel Morland (1625 – 1695), invented three different kinds of calculating instruments.
On a visit in 1653 to the court of Queen Christina of Sweden, Morland got to see a Pascal's adding machine (see chapter 14) which Pascal had, in 1652, sent as a gift to the Queen.
In a small book, published in 1673 *The Description and Use of Two Arithmetic Instruments* Morland described two different kinds of his calculating instruments:

- a simple adding machine, see Figure 12.4 and
- a mechanical version of Napier's bones,

see Figure 12.5, both invented in the middle 1660s.
The third device combined the basic sector with some simple gearing to produce a robust trigonometrically calculating machine.
The adding machine had no proper carry mechanism. Instead, a number of small auxiliary dials were constructed above each of the major dials and, every time a major dial was turned from 9 through 0, a small tooth would advance the auxiliary dial by one position.

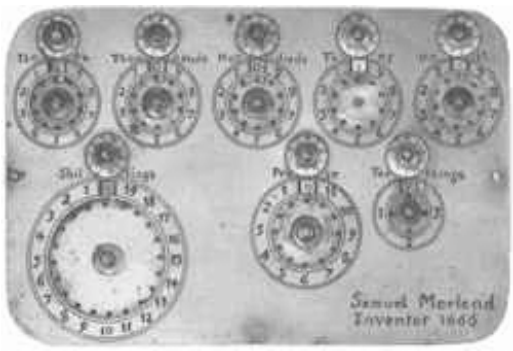
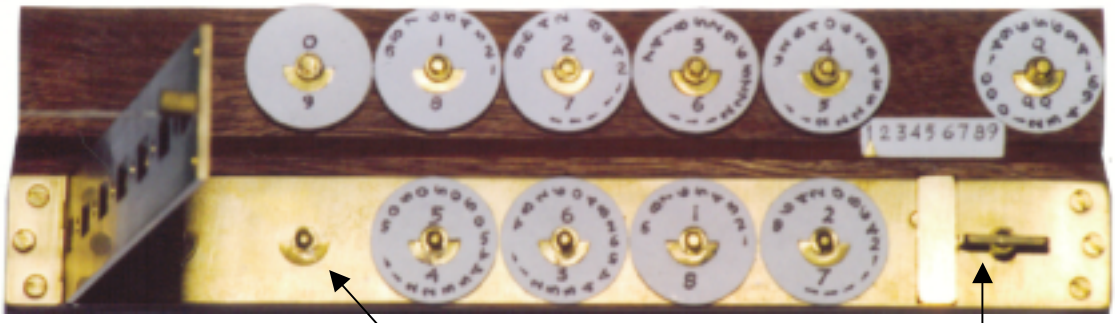


Figure 12.4

12.5 Morland's mechanical version of Napier's bones

This calculator was designed as an aid to multiplication, division, and extracting root using the principles involved in Napier's bones. It consisted of a flat box, covered with a brass plate, with 2 series of vertical pins. The lower seven pins were connected to each other through an elementary rack-and-pinion mechanism inside the box. The right-most pin was provided with a key. The act of turning the key would move the rack and rotated the six semi-circular pins an identical angle and simultaneous moved the pointer along the multiplier indicator.

5 Disks for multiplication and 1 for extracting roots



perforated, hinged lid (upright) 6 semicircular pins key

On the upper series of 6 pins flat disks were stored. The calculator came complete with 30 disks for multiplication and division and 5 special disks marked with the letters Q / QQ, used for finding square and cube roots, stored on the right-most pin.

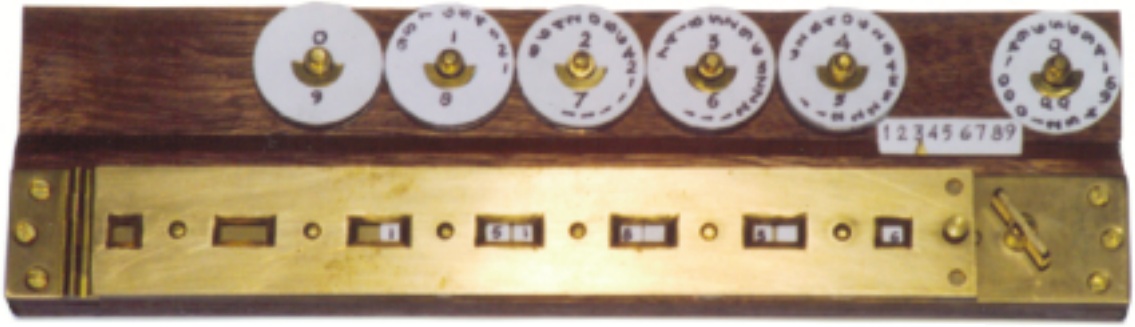


Figure 12.5 added 1 6 8 3 6

These disks were simply a circular version of Napier's bones with the products being set down around the edge of the disk in such a way that the two parts of a product would be placed at the opposite ends of a diagonal.

For example, Figure 12.6 is one of his disks showing his version of Napier's bone for the number 3; the product of 4 x 3 = 12 is shown highlighted by arrows and adjacent digits represent the product 3 x 3 = 9 and 5 x 3 = 15.

The large digits engraved in the middle of the disk indicate which bone is being represented; the upper digit identifies this side of the disk and the lower digit indicates which bone can be found engraved on the underside.

Figure 12.6

A hinged perforated lid could be opened for placing the proper disks on the proper places for the multiplicand, for instance, the disks marked 5, 6, 1, and 2. After closing the lid, the key can be turned till the pointer indicates the multiplier, in this case 3. Than the partial products can be read off. Add the two digits in each window: 1, 5 + 1, 8 + 0, 3 + 0 and 6 or 16836. The advantage compared to cylinders is that the units and tens, which belong together and have to add, now are next to each other.

12.6 Schickard's calculator used for fractions up to 7 decimals

Around 1600 the astronomers had to calculate with the natural values of sine and cosine. They did not use the radius $r = 1$, but set $r = 100{,}000$ or $1{,}000{,}000$. So they dealt with large numbers, sometimes up to more than 10 digits.
Jost Bürgi, (see § 9.5) as well as Henry Briggs, (see § 9.7) used only the first 7 figures in calculation with such large numbers to avoid needless work, and as they only considered these as important.
Multiplication of numbers of up to two 6 place decimals is possible in using Schickard's calculator. The method is called *multiplication from left to right*. The next example of multiplying decimal fractions up to 6 digits and get results with up to 7 decimals and taking the 8th decimal into consideration: .123456 x .987654

```
123456  input of multiplicand

 0| 9        |  push slide 9 to the left (9 x 1; then 9 x 2;  9 x 3 to 9 x 6)
o | 18       |  bell rings    If in the operated example of § 12.3 instead of the 5612 and 37 larger
  |  27      |  numbers were selected, the result could exist of 7 figures. If at some time the
  |   36     |  left-most figure of the adder is 9, the tens carry going out. Schickard solved this
  |    45    |  problem by letting a bell ring operated by the dark tooth of the 6th wheel. To
  |     54   |  retain that 1, Schickard used a ring which he placed on one finger, indicated by a
                small o.

o | 111104   |  first intermediate result, dial 9 into highest memory position
  | 08       |  retract slide 9 and push slide 8 to the left
  |  16      |
  |   24     |
  |    32    |
  |     40   |
  |      4   |  8
 o| 209868   |  add 1 for rounding

 o| 209869   |  second intermediate result, dial 8 into next position of the memory unit
  |  07      |  retract slide 8 and push slide 7 to the left
  |   14     |
  |    21    |
  |     28   |
  |      3   |  5
  |          |  42
 o| 218510   |  add 1 for rounding

 o| 218511   |  third intermediate result, dial 7 into next position of the memory unit
  |  06      |  retract slide 7 and push slide 6 to the left
  |   12     |
  |    18    |
  |     2    |  4
  |          |  30
  |          |   36
 o| 219250   |  add 1 for rounding
```

```
o|  219251|   fourth intermediate result, dial 6 into next position of the memory unit
 |       05 |   retract slide 6 and push slide 5 to the left
 |        10|
 |         1|  5
 |          |  20
 |          |    25
 |          |      30
o|  219312|   add 1 for rounding

o|  219313|   fifth intermediate result, dial 5 into next position of the memory unit
 |        04|   retract slide 5 and push slide 4 to the left
 |         0|  8
 |          |  12
 |          |    16
 |          |      20
 |          |        24
o|  219317|   add 1 for rounding
.1219318   result
.121931812224 the fully computed result
```

Remember, the small ring (o) stands for 1(outside the calculator).
Note that there are two possibilities to take the rounding into consideration:
- direct rounding as done in the example;
- cumulated rounding. That is, placing rings on another finger and adding the number of rings to the result.

In the next chapter a very different form of calculating rods will be explained.

1) Michael R. Williams
A History of Computing Technology

2) Friedrich W.Kistermann
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Chapter thirteen, *Genaille - Lucas rulers*

At the French Association for the Advancement of Science meeting in 1885, Eduard Lucas presented a problem on arithmetic which caught the attention of Henri Genaille, a French civil engineer working for the railway. Genaille, who was already quite well known for his invention of several different arithmetic aids, solved Lucas' problem, and in the process, devised a different form of Napier's bones. These rulers eliminated the need to carry digits from one column to the next when reading off partial products. He demonstrated these rulers to the Association at their meeting in 1891.

13.1 The rulers of Genaille for multiplication

These rulers a set of which are shown in Figure 13.1 are similar in their use to a standard set of Napier's bones.
There is one ruler for each digit from 0 to 9, a set should contain as many of each ruler as desired.
Each ruler is divided into nine sections with several digits inscribed in each section, and one or two arrows pointing to the left toward a particular digit in the next ruler.
At the left there is always the need for having a leading zero ruler, at the right a ruler with the digits from 1 to 9.
Figure 13.2 illustrates how to find the product of 3271 x 4.

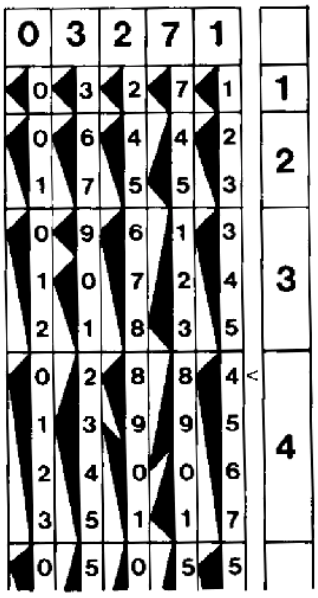


Figure 13.2

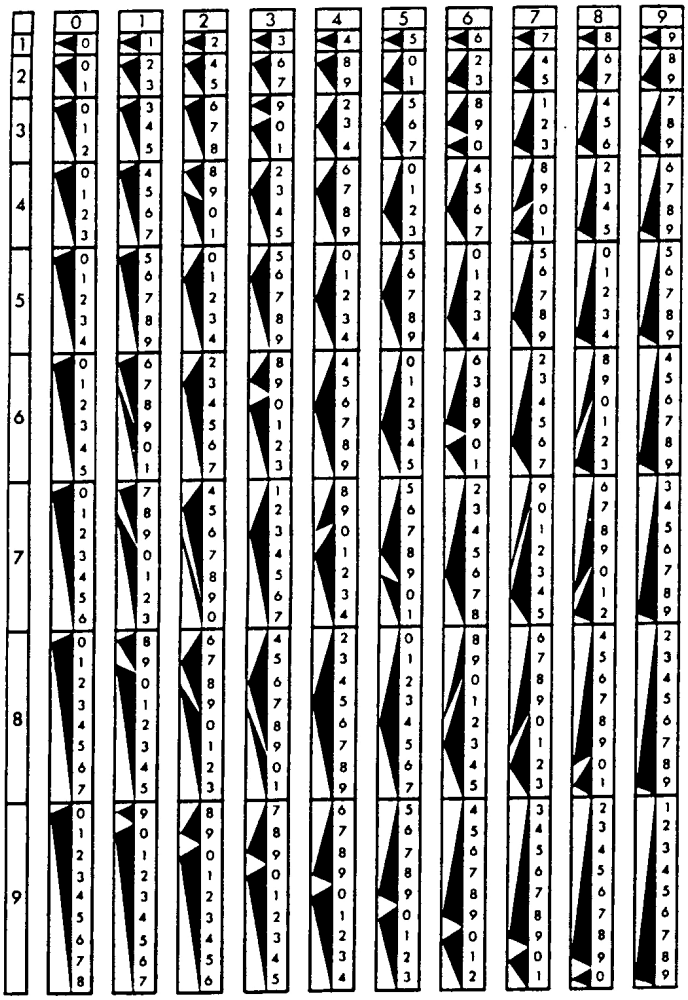


Figure 13.1

The rulers for 0, 3, 2, 7 and 1 + the ruler 1 to 9 are placed side by side. Starting with the fourth section (because the multiplier is 4) of the most right ruler, you select the digit at the top of this section (4 in this case, see <) and than simply follow the arrows toward the left, reading off as you come to them – in this case 13,084

13.2 The rulers of Genaille for division

The division rulers are similar to the multiplication ones, a set of which are shown in Figure 13.3. There is one ruler for each digit from 0 to 9, a set should contain as many of each ruler as desired. Each ruler is divided into nine sections with several digits inscribed in each section.

The large arrows are replaced by a multitude of lines at the right side of the digits, pointing to the right toward a particular digit in the next ruler.

Together with a special ruler marked R to be placed on the right-hand side of the set in order to determine the remainder (if any) of the division operation and one ruler with the digits from 2 to 9.

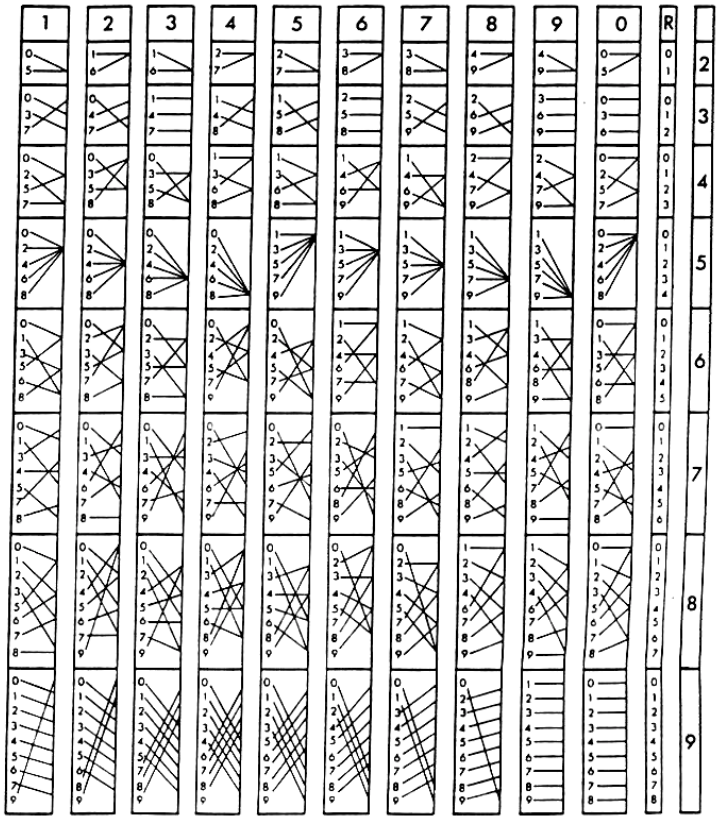


Figure 13.3

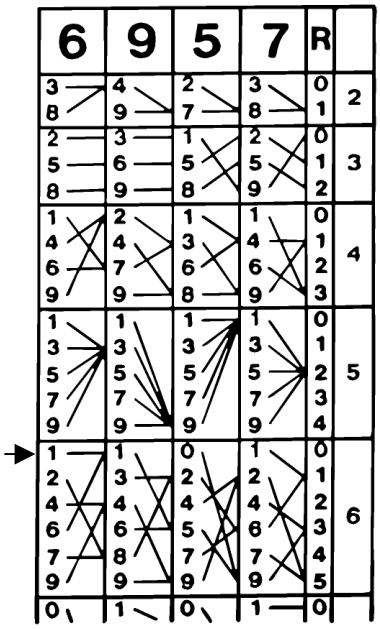


Figure 13.4

Figure 13.4 illustrates how to find the quotient of 6,957 by 6
The rulers for 6, 9, 5, 7 and R + the ruler 2 to 9 are placed side by side.
Starting with section 6 (because the divisor is 6) with the topmost digit (1 in the case shown here) see ➔ , simply follow the lines toward the right and reading off the digits as they are encountered: quotient 1,159 with a remainder of 3.

6957 ÷ 5 = 1391, remainder 2

6957 ÷ 4 = 1739, remainder 1 and so on.

Chapter fourteen, *Mechanical hand-driven calculators*

14.1 Addiators

14.1.1 Blaise Pascal and his Pascaline [1)]

When Blaise Pascal (1623 – 1662) was at the age of four years, his mother died. Father Etienne Pascal, a reasonable mathematician, sold his high officially function at Clermont in Southern France and moved to Paris. Blaise did not attend school, but was educated entirely by his father, whose library served as the boy's schoolroom. After having been taught elementary skills of reading and writing, Blaise was essentially left to develop his own education by reading through his father's library. Blaise, like most boys who have been told not to read certain books, made a point of giving up his playtime to secretly read mathematical works. After a serious disagreement with Cardinal Richelieu, Etienne was sent off to Rouen and given the job of "Commissioner for the Imposition and Collection of taxes in Upper Normandy." The daily grind of adding up long columns of tax figures gave Blaise the impetus to design something, which would relieve some of the tedium.
When Blaise was only 19 years old, he managed to design the first of his many calculating machines. He hired a local workman and, showing him his carefully done drawings, asked him if he could make the instrument. But what he produced was quite unworkable. This led Blaise to train himself as a mechanic and learn the basics of constructing metal parts.

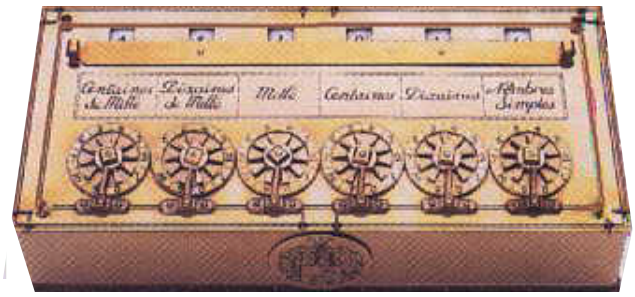


Figure 14.1

In 1642 he succeeded to produce his first adding machine, the *Pascaline*, See Figure 14.1. The Pascaline was made of metal and equipped with 5 up to 10 dials. The machine often was made to figure French currency, the two right-hand dials are numbered for sou and denier while the other dials to the left are numbered from 1 to 9 and 0 for livre. 1 livre = 20 sous; 1 sou = 12 deniers.

To add a number, say 3, it was only necessary to insert a small stylus into a dial at the position marked 3 and rotate the dial clockwise until the stylus encountered the fixed stop. The result could be read on revolving drums through windows at the top of the Pascaline. A disadvantage was that the dials could only turn clockwise. Subtraction could be performed by adding the *nines complement* of the number, see example. Therefore the drums were provided with two series of figures, see Figure 14.2; under each figure for addition is engraved the complement of 9.
A brass slide could cover the lower part of the windows for normal addition and the upper part for performing subtraction. Example: 5612 - 1937

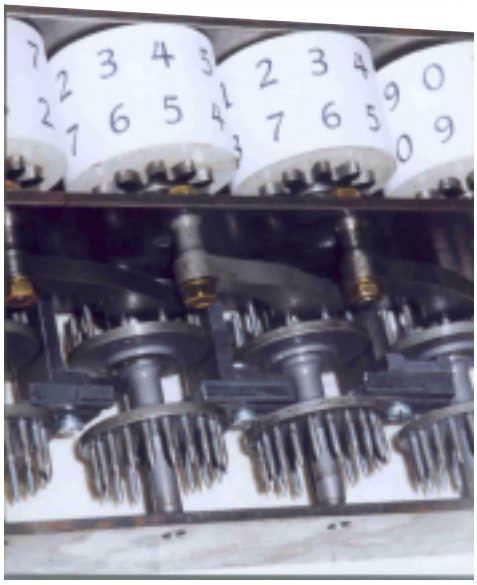


Figure 14.2

5612	5612	first add to 5612 the
- 1937	+ 8062	complement 9999 – 1937 = 8062
3675	13674	than transpose the 1
		3674 + 1 = 3675

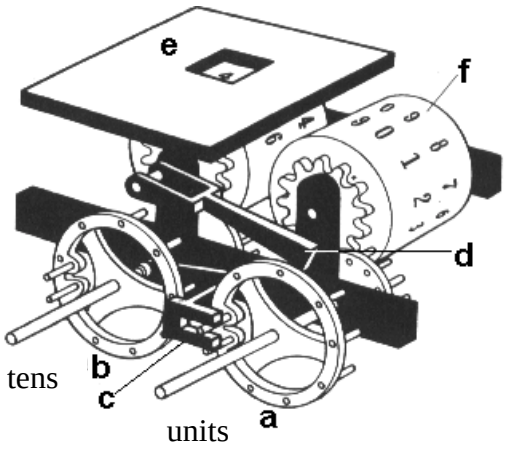


Figure 14.3

The carry mechanism of Schickard's calculator (see § 12.3) operated with only one tooth upon which strong forces acted if more carries coincided, e.g. 3999 + 1. Different from that mechanism, Pascal designed a mechanism that was based upon falling weights rather than a long chain of gears, as shown in Figure 14.3.
On the backside of wheel marked **a** 10 pins are attached between which the catch **d** falls. This catch is fixing wheel **a** to prevent unintended movement, but also due to this catch, it is not possible to turn the dials counter-clockwise in order to do subtraction.
On the front-side of the same wheel **a** 2 pins are attached which lift up the weight **c** if wheel **a** rotates. When the wheel rotates from 9 to 0, the pins slip out of the weight, allowing it to fall, and in the process the little spring-loaded foot, attached to **c**, will kick at the pins sticking out of wheel **b**, driving it around one place. This gravity-assisted carry mechanism was placed between each pair of digits in the accumulator and, when a carry was generated through several digits, could be heard 'clunking' all the way down the line.

Although Blaise produced about 50 different machines during his short lifetime, they were all based on the idea incorporated in his first machine of 1642. Only Blaise himself could repair the device and it cost more than the people it replaced! In addition, the first signs of technophobia emerged with mathematicians fearing the loss of their jobs due to progress.
For centuries one took the Pascaline for the first invented adding machine. After the discovery of the letters and drawings from Schickard to Kepler (see § 12.3) we know better.
Pascal's triangle or the binomial $(a+b)^m$ was already known from the 9th century. See also the chapters about China and Japan.

14.1.2 The Addometer

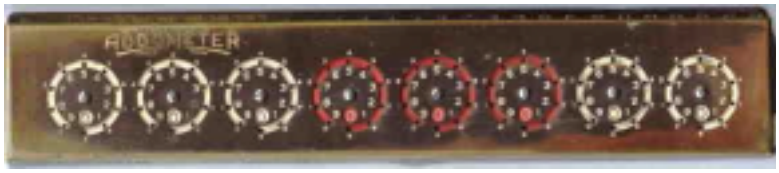


Figure 14.4

The use of dials for entering numbers into an adding device was in practice up to the 20th century. A well known of such a calculator is the Addometer, see Figure 14.4.

14.1.3 The Comptometer [1)]

Not before the coming of the *comptometer*, a key-driven adding machine from Dorr E. Felt in 1885, see Figure 14.5, was mechanically adding fast, easy, and reliable.
Dorr Eugene Felt (1862 - 1930) grew up on his parents' farm and attended only a few years the High School. His first job was as a mechanic in a metal-factory. In 1882 he moved to Chicago, where he worked also as a mechanic.

Figure 14.5

There, while running a planer, he conceived an idea watching its ratchet feed motion. The motion, which was like that to be found on all planing machines, could be adjusted to ratchet one, two, three, four or more teeth for a fine or coarse feed. Felt, not being able to afford the cost of having a metal model constructed, decided to make one of wood. As he tells the story:
I went to the grocery and bought a macaroni box to make a frame of. I went to the butcher and bought skewers to make the keys of, and to the hardware store and bought staples, and to

the bookstore and bought rubber bands to use for springs. I went to work to make a calculating machine. I began on Thanksgiving Day (1884) because it was a holiday, and worked that day, and Christmas and New Year's, but I didn't get it done in three days. It was a long time before I got it done.
First he made his 'macaroni box' (that still exists), finished it soon after New Year's day, 1885 and managed to produce the first fully working comptometer in the autumn of 1886.

Figure 14.6 The interior-work

On his machine one could enter figures one after another with one hand and simultaneous slide with the other hand along the paper work which made it possible to work very fast. Only after the last enter, and after writing down the result, the machine could be 'reset' by pulling the reset lever on the right side of the machine. Till now all efforts to construct a good working adding machine got stuck by the same problems.

The first problem arose when a key was pushed to fast. The cylinder with figures on which the result has to read off has to stop revolving at the right moment.
The second arose when two or more keys were pushed at the same time and the machine gets stuck: the carries had to be done one after each other. Felt solved both problems in an ingenious way, making it possible to perform a multiplication on the comptometer, for instance: 5612 x 37. The operator has to put fingers on both the 3 key and the 7 key (in the tens and units columns, respectively) and pushes then both at the same time 2 times, then moves the hand one position to the left (in the hundreds and tens columns), pushes 1 time, and so on.

14.2 Four Function Calculators

14.2.1 Stepped Drum Machines

14.2.1.1 Gottfried Wilhelm Freiherr von Leibniz [2)]

Gottfried Leibniz (1646 – 1716) was the designer of an extended adding machine with tens carry, not a genuine multiplying device. See Figure 14.7.
When Gottfried was at the age of 6 years his father died. Young Gottfried (who by that time had learned some Latin) was allowed to have free run of his father's library. At the age of 15 he entered the University of Leipzig to study law and, when he was 20, he applied for the degree of Doctor of Laws. Gottfried took a job as an advisor to the Elector of Mainz. While he was in his service, Leibniz travelled extensively through Europe and actually resided in Paris for several years. It is known that, when Leibniz heard Pascal had invented a mechanical adding machine, he wrote to a friend in Paris asking for details of its construction. In Liebniz's notes is a series of suggestions and drawings for an attachment to be placed on top of Pascal's device in order to enable it to perform multiplication. It never was brought in practice. Leibniz made a statement: "It is beneath the dignity of excellent men to waste their time in calculation when any peasant could do the work just as accurately with the aid of a machine". We know that Leibniz started to think about the problems involved in designing a multiplication machine sometimes about 1671. In January of 1672 he happened to be in London and was able to demonstrate a wooden model (which did not work properly) to the members of the Royal Society. The Secretary of the Royal Society did not invite Leibniz to the next meeting but suggested that, when a proper working model was available, they would like to have it demonstrated. The same types of problems that were faced by Pascal and others – poor workmenship and poor materials, plagued Leibniz. The final machine was only put

together because Leibniz had found, during his stay in Paris, a French clock maker named Olivier, who was both honest and a fine craftsman. In 1674 Leibniz wrote a manuscript entitled "Ma machine arithmetique de la maniere que je l'ay fait faire a Paris l'an 1674."
In 1712 Leibniz sent a calculator to Gottfried Teuber at Z eitz (Germany) to have this machine completed. The machine that Teuber had has obviously vanished without leaving a trace.
In 1764 another calculator was sent to Abraham Gotthelf Kä stner for the same reason. Leibniz published very little about his machine, and even that he did very late. In 1710 Leibniz published a drawing that shows the exterior of his calculator, see Figure 14.7, accompanied by operating instructions. Nothing was revealed about its internal mechanism, let alone the stepped wheel.

Figure 14.7

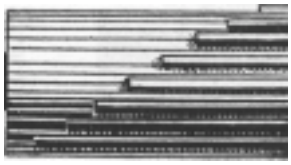


Figure 14.8

The centre of Leibniz' machine was the stepped wheel, a drum with 9 teeth of different lengths, gradually increasing from one end of the series toward the other end, and are arranged over ca. a third of its circumference in an arc about the axis of the driving drum, with which they are parallel, see Figure 14.8.
To perform a multiplication, the machine had to be composed of 2 parts: an eight-place setting mechanism, the *carriage*, at the front side, that may be shifted with the aid of a screw spindle and a crank on the left side and a stationary calculating mechanism with sixteen places at the back side, see Figure 14.7. In order to be able to multiply by a multiplier > 9, the setting part had to be shifted one or more 'decimal' places. The multiplicand had to be set by turning one or more of the 8 dials **d** at the top of the carriage, (Figure 14.9). Those dials could through a pinion-and-rack connection **p** let shift an axis **a** a distance dependent of the turning of the dial in accordance with the digit of the multiplicand. On each axis were fixed a narrow gear and a stepped drum. The narrow gear would interact with a wide gear that let the narrow gear slide without losing contact. Both gears are coupled with similar gears of the other 7 sets.

The stepped drum would interact with a different number of teeth with the next gear **g**, which would turn the result wheel.
Leibniz' ten carry mechanism was ingenious.
An example of a multiplication: 5612 x 37.
First set 7 on the disk **s** to the right of the dials **d** to stop the operation when the crank on the front of the carriage has turned 7 times. In this way, the multiplication by 7 is made by 7 repeated additions.
Than the carriage has to be moved 1 place to the left by help of the crank on the left side of the machine. Than the next figure of the multiplier has to be operated on the same way.
During moving the carriage, between 2 operations, the teeth of the stepped drum are on the bottom side in order not to change the result dials.

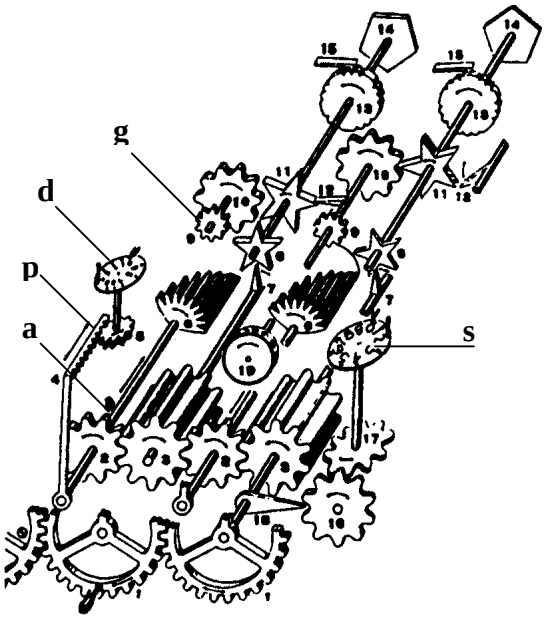


Figure 14.9

The idea was very good, however by poor workmenship and poor materials, it was not possible to construct a machine, which worked faultless, in spite of a fortune Leibniz had put in it.

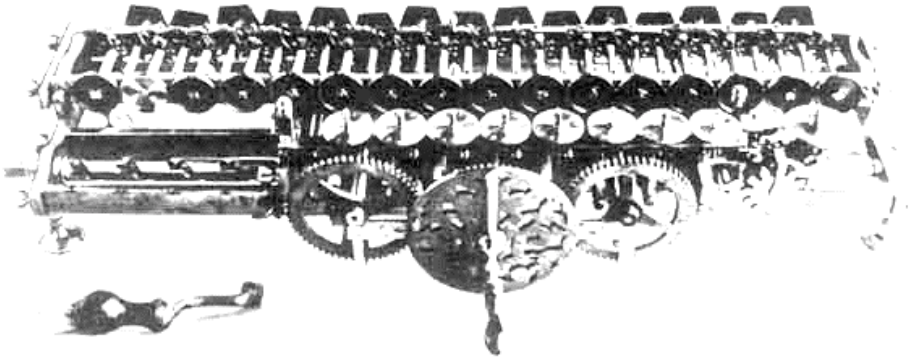


Figure 14.10 The internal mechanism of Leibniz' calculator

The calculator that was sent to Kä stner in 1764 was rediscovered in 1876 in Göttingen (Germany) by J.B. Listing. Non of Kä stner's contemporaries was interested in this machine (though the existence of a calculating machine in Göttingen has been reported at various times). This machine was transferred to Hannover in 1879 by Bodemann. Wilhelm Jordan studied the Leibniz calculator from 1887 until 1898. He arranged for Burkhardt, who was the manufacturer of the Arithmometer at Glashütte, to see the Leibniz calculator at Hannover, and Jordan persuaded him to restore this machine. The calculator was shipped to Burkhardt in March 1894. In October 1896, the restoration was completed (dimensions 67 x 27 x 17 cm). Now there are more copies, (from ca. 1890), made by Grimme, Natalis & Co A.G. of which one in the Landesbibliothek in Hannover. (See Odhner, § 14.3.2)
In 1679 Leibniz published *De Progressione Dyadica* in which he described his binary system. Napier already used the binary system in his Location Arithmetic as Performed on a Chessboard.

14.2.1.2 Circular Stepped Drum Machines

Philipp Mathäus Hahn[3,4,5)](1739 – 1790), son of a clergyman, later also became a clergyman himself at Kornwertheim. He practised astronomy and he constructed perfect astronomical watches. For performing calculations for designing parts, he liked to build a calculator.

Figure 14.11

He was well informed on as well the financial as the technical aspects of Leibniz' problems, but in spite of that, he used a variant of the stepped drum in his originally design. After years he managed to produce an excellent working machine in 1778, of which he had much help. See Figure Figure 14.11. The machines existed of a cylindrical house. On the circumference of its top were 11 knobs with stepped drum-like parts which were set by pulling them up.
To be able to continu to spend his time on astronomy he let more copies be constructed by his brother in law Johan Christian Schuster, who was watch maker in Uffenheim. However, those copies were rather expensive.
The inner small disks on the top of the machine counted the rotations, the multiplier; the larger ones showed the result. The outmost figures of these disks are black for addition and multiplication;

the inmost figures are red for subtraction and division and are in reversed order (complement). The tens carry mechanism is less complicated because the stepped drums were operated one after each other. Hahn deserves the credit for having designed the first practical-ly usable machine for the four fundamental operations of arithmetic. One of Hahn's machines may be found in the Deutsches Museum in Munich.

Figure 14.12

In 1784, **J ohan HelfreichMüller**[4)] designed a calculation machine, which he had built by a clock maker from Giessen. It is very similar to Hahn's machine, but Hahn's setting rods, which were adjustable in height and had to be handled very carefully, were replaced by revolving disks. See Figure 14.12.
Hahn's brother-in-law manufactured calculating machines embodying Müller's improvements. One of Müller's machines may be found in a museum in Darmstadt.

14.2.1.3 A variant on Circular Stepped Drum Machines [6)]

Christian Hamann (1871 – 1948) constructed a calculating machine (known as Gauss) based on the stepped drum. His machine possesses only a single actuating element in the form of a developed stepped drum that, by rotation of the crank, is moved past, or brought into mesh with, numeral gears positioned in a radial direction around the shaft of the crank, see Figure 14.13. The entering of the value occurs with the aid of setting slides that move in seven slots which are likewise radially located relative to the shaft of the crank. Below each slot is a square shaft. On each square shaft he adjusted a gear which could be moved by setting knobs along a distance indicated from 1 to 9. Dependent on its place along the slot, the gear, and a result dial, with white and red digits, made a part of a full turn when the disk with the teeth made a full turn operated by the crank. A tens carry mechanism (only one is drawn) completed the calculator. The weight of the machine is only 850 grams; it has a diameter of 12.5 cm and is 10 cm high.

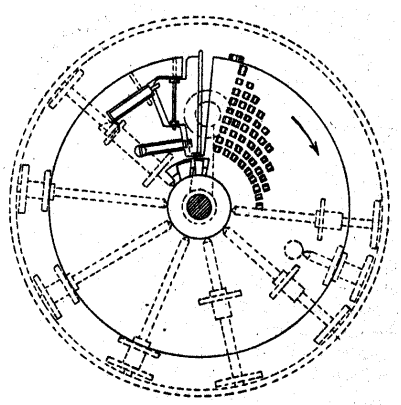


Figure 14.13

14.2.1.4 The Arithmometer [1)]

Charles Xavier Thomas from Colmar (Alsace) (1785 - 1870) was founder and manager of the Compagnie d'Assurance Le Phé nix in Paris. Nearly 150 year after the invention of Leibniz' stepped drum Thomas obtained in 1820 patent for his design of a calculator which became known by the name of *Arithmometer* especially in France or as the Thomas machine. Although based on the Leibniz' stepped drum gears, the Arithmometer incorporated several features which made it easier to use. For example, Thomas reversed the operating motion (during a multiplication operation) of the result registers, so that the result registers moved to the left over the top of the fixed stepped-gear mechanism rather than attempting to move the stepped-gear mechanism under the result register.
These changes, together with the improvements introduced because of better manufacturing techniques for the gears, resulted in a machine which was basically reliable in operation even if some models were large enough to cover a complete desk and often required two men to move them from place to place. The fixed part of the machine which contains the setting mechanism consists of the cover or setting plate with the setting slides side by side in slots

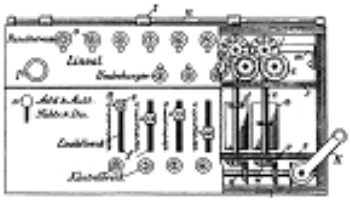


Figure 14.14

above the setting gears. These gears may be moved along square shaft and are connected by a fork to the setting slides. They are capable of sliding back and forth to engage different numbers of teeth disposed over about a third of the stepped drum. The result mechanism contains the result dials and the revolution counter dials. See Figure 14.14.
The crank on the right was always turned clockwise whether it performed addition (multiplication is repeated addition) or subtraction depended on the setting of the lever on the far left. Originally the machine was furnished in a wooden chest having a lid with lock and key, as Figure 14.15. Later they were usually built into a metal frame with legs. Up to the time when the calculator machine industry was introduced into Germany by Arthur Burkhardt in 1878, the Thomas workshop was the only firm in this line and supplied the whole world with its products. Till 1878 ca. 1500 Arithmometer's were produced.

Figure 14.15 Burkhardt's machine

Like every successful product, the Thomas Machine had many clones. An example is the improved Arithmometer introduced by **Arthur Burkhardt** in 1878, see Figure 14.16, one of Burkhardt's early models, made in wooden boxes. His company, The Erste Glashütter Rechenmaschinenfabrik (Sachsen) is credited with laying the foundation of the German calculator industry. Other machines based on the Thomas design is the **MADAS** (Multiplication, Automatic Division, Addition, and Subtraction) introduced in 1913, manufactured by H.W. Egli of Switzerland. The **TIM** (Time is Money) introduced by Ludwig Spitz in 1907. Also the Archimedes, Austria, Saxiona, Ideal, Peerless, Unitas, Bunzel-Delton. These calculators were used until 1950 here and there.

14.2.1.5 The CURTA

Curt Herzstark (1902 – 1988), was the son of a businessman in Vienna. Father Herzstark founded in 1905 the company 'Rechenmaschinenwerk AUSTRIA Herzstark & Co.', in Vienna, which built different varieties of the Thomas-calculators with additional patented Herzstark-inventions, see Figure 14.16.
In 1916, after his 'Realgymnasium' Curt started an apprenticeship as precision mechanic and toolmaker in the factory of his father.

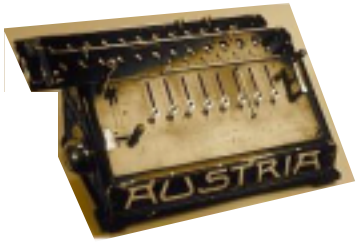


Figure 14.16

After finishing his final test 'with honours' he started studying engineering at the 'Höhere Staatsgewerbeschule' in Vienna. Curt did internships in his father's factory, where he worked in Assembly and Sales. Curt got the idea for the construction of a four-function pocket calculator from remarks made by customers during his sales travels.
In the winter of 1937/38, after years of tinkering, the construction of a four-function calculator was essentially completed, and a first prototype was built and working. In March 1938, Austria was 'brought back home' in the Reich by the Nazi's. As a consequence, the German army surveyed the industry of the annexed country, so they also examined the Austria-factories. No production was started; from the summer of 1938 on the Austria-factories had to make measuring devices for the German army. On the other hand, Curt didn't want to make his invention public for reasons of competition.

In May 1943 Curt was arrested by the nazi's. In the fall of 1943 he was deported to concentration camp Buchenwald. To his surprise the camp commander was already informed about his work and ordered him to make drawings of the construction. He wanted to give the machine to the Führer after the successful end of the war. Until the liberation of Buchenwald on April 11, 1945 by the Americans, he had redrawn the complete construction from memory.

There were many candidates who wanted to produce the machine. Prince Franz Josef II of Lichtenstein had the best offer. He invited Curt to build a factory for the production of the calculator in Mauren. And so the Contina AG was founded in 1946. The first calculators named CURTA were of a model with 8 x 6 x 11 positions, in 1954 model II with 11 x 8 x 15 positions was added. The production of calculators was stopped in 1972. Until that date, 80,000 I's and somewhat more than 61,000 CURTA II's had been made. The largest diameter of the CURTA I has only 53 mm and its largest height is 77 mm, see Figure 14.17; CURTA II, see Figure 14.18, respectively 65 and 81 mm, so easily to be held in one hand.

Figure 14.17

Figure 14.18

On top is the operation handle, which only is allowed to be turned clockwise. The upper knurled ring of the CURTA is the carriage, see Figure 14.18. To move the carriage from one position to another, the handle has to be in its zero stop, than raise the carriage straight upward, and turn it until the indicator arrow points to the required position and let the carriage down into its lower position.

The stepped drum is composed of disks with pins, see Figure 14.19. To make it possible to subtract and to divide, there is a second series of disks, a complement stepped drum. In its zero stop position the handle can be pulled upward to use that second part for subtractive (minus) turns (clockwise). The 8 (or 11) setting knobs are numbered at the bottom of the setting register. Moving a knob downward along its screw-shaped axis, any number from 1 to 9 can be set and checked in the corresponding setting dial slot above. In Figure 14.19 only one of the setting knobs is represented.

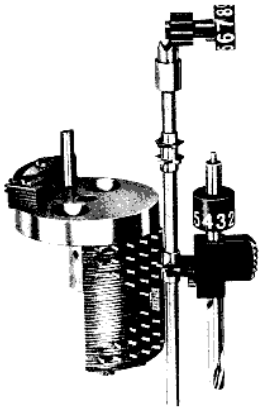


Figure 14.19

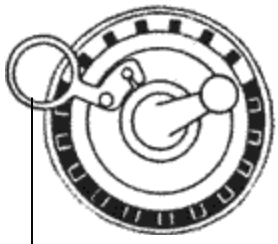


Figure 14.20

On top a result dial is shown.
The counting dials are located in a circle around the top of the machine. The black dial is the result or product dial, the white dial the counter or quotient dial. Clearing the dials is done with the clearing lever ring, which has two stop positions. To clear both dials, raise the carriage and move the clearing lever steadily around the dials, bringing it back to the initial stop position.

Example of a multiplication 5612 x 37:
Set register with the knobs 5612; turn handle 7 times; set carriage in position 2; turn handle 3 times; read result.
Or: consider 37 as 40 - 3: first set carriage in position 2; turn handle 4 times; set carriage in position 1; pull handle upward and turn 3 times (clockwise).
Dividing can be done by the 'Egyptian' method: example 729 ÷ 32,4 or: what is the number to multiply 32.4 with to obtain 729?

Set 324 in the register and set the carriage in the 6th position. After 2 rotations of the handle the result is 648…(result after the third rotation would be > 729)
Set carriage in the 5th position and make two rotations, result is 712.8…
Finally set the carriage in the 4th position and make 5 rotations: result 729.000
Read the quotient: 22.5.

Extracting square root on the CURTA

Example $\sqrt[2]{1369}$			read on counting
Obviously the root has 2 figures, therefore set the carriage in the second position:			dial:
set second knob on	1,	turn handle 1 x clockwise;	100
set the same knob on	3,	turn handle 1 x clockwise;	400
set the same knob on	5,	turn handle 1 x clockwise;	900
one more rotation would give a result > 1369			
now set the same knob on 6 and set the carriage in the first position;			
set the first knob on	1,	turn handle 1 x clockwise;	961
set the same knob on	3,	turn handle 1 x clockwise;	1024
set the same knob on	5,	turn handle 1 x clockwise;	1089
set the same knob on	7,	turn handle 1 x clockwise;	1156
set the same knob on	9,	turn handle 1 x; clockwise	1225
set the same knob on	11,	that means: 6 becomes 7 and 9 becomes 1;	
		turn handle 1 x clockwise;	1296
set the same knob on	3,	turn handle 1 x clockwise;	1369
finally read the root on the rotation counter (= product window): 37			

14.3 Variable Tooth Gear Machines

14.3.1 Poleni [4,5)]

Already in 1709 **Giovanni Poleni**, professor of astronomy and mathematics at the University of Padua, published a treatise *Miscellanea*. In *Miscellanea* he described a variable tooth gear mechanism. He constructed a prototype of a calculator based on that mechanism and driven by weights.
The idea was ingenious, but because neither the workmenship nor the materials were available to implement the idea it did not work properly. Poleni himself destroyed his calculator.
Recently IBM in Italy built a model, see Figure 14.21.

Figure 14.21

14.3.2 Antonius Braun [3)]

Figure 14.22

In 1727 Antonius Braun (1685 - 1728), an instrument maker in Vienna, constructed a circular calculator with, instead of one or more stepped drums as Hahn's calculator, one concentric disk provided with 6 radially sets of 9 parallel variable tooth. To set the multiplicand 1 to 9 of the tooth of each of the 6 radially sets have to protrude. After the multiplicand was set, the disk was turned by a crank as many times as the multiplier requered to rotate the counter wheels.
Figure 14.22 is Braun's calculator based on stepped drums.

14.3.3 Baldwin / Odhner / Brunsviga

Frank Stephan Baldwin (1838 - 1925) patented in 1875 the Baldwin principle in which the stepped drum is replaced by sets of pinwheels. The pinwheel consisted of a main wheel body which was fixed relatively to the shaft and contained nine radial pockets large enough to allow nine moving pegs to protrude or retract completely into it. A thin disk with the number setting lever was placed over the main disk. This disk held the pegs in place and could be rotated relatively to the main disk. This rotation controlled the position of the pegs. The covering disk has a cut-out channel. The pegs have a protrusion that fit into the open channel in the covering disk. By adjusting the lever, the covering disk rotated, the number of pegs that stuck out was controlled. After the lever was set, the crank was turned to rotate all the wheels. This caused them to rotate the counter wheels by the appropriate amounts. One set of pegs is used for each digit. For Odhner's pinwheels, see Figure 14.23. Between 1873 and1913 Baldwin made several models; from 1912 he started with Jay Randolfh Monroe the Monroe Calculating Machine Company.

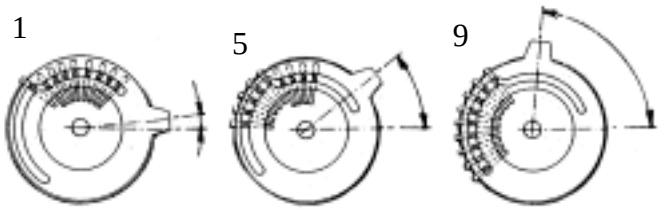


Figure 14.23

Willgodt Theophil Odhner (1845 - 1905). He left his home in Vä rmland, Sweden, at the age of twelve and went to Stockholm, originally with the intention of becoming a merchant. Even at that time his mind was set on technical matters and soon he recognised his true calling. He thus left the counter and entered a machine factory where he worked for several years gaining practical experience. He also prepared himself, by diligent study, for entry into the technical college. Having completed a three-year course, he obtained employment in the St. Petersburg plant of the Swedish industrial tycoon Alfred Nobel, but was later appointed to manage the Imperial Russian workshops. There he developed his calculator based on the variable tooth gear. Odhner patented his pinwheel device in 1878. He started production in St. Petersburg ca. 1886. The calculators were sold under the name Original Odhner. The actual mechanism was very similar to that of Baldwin, and the machines constructed from them were known by the generic term *Baldwin Machines* in America and *Odhner Machines* in Europe.
The large crank on the right was turned clockwise for addition and repeated turned for multiplication and in the opposite direction for subtraction and division. The number to add or the multiplicand was entered with the levers on the top (the actual number entered of Figure 14.24 is 5264302505) and the result appeared in the window on the right of the carriage below. The small windows on the left on the carriage counted the number of turns of the main crank is the multiplier. The small crank on the right of the carriage cleared the results windows and the crank on the left cleared the turn counter. The levers below the carriage allowed it to move freely in either direction: to multiply with the second figure of the multiplier the carriage had to move one place to the right. Dividing was performed by entering the dividend with the levers, than entering the dividend in the window to the right by one turn of the crank; after entering the divisor in the machine with the levers, the division could be performed by turning the crank counter clock-wise.

Figure 14.24

Odhner's patent of 1878 was not in the name of Odhner but in the name of Köningsberger and Co. of St. Petersburg and Odhner ceded the German manufacturing rights to that company. This company, however, did not go into production and sold their rights in 1892 to Grimme, Natalis & Co of Braunschweig, who after some improvements have very successfully distributed the machine, under the name of *Brunsviga*, all over the globe.

During the Revolution of 1918 the factory in St. Petersburg became the property of the state, the equipment moved to Moscow. Odhner's legal successors went to Sweden where they founded the Aktiebolaget Original Odhner in Göteborg. Later the machines were named Facit. Many manufacturers used the Odhner system up to the 70s of the 20th century, among others: Arithmos, Berolina, Chateau, Felix, Lupsia, Melissa, Rapid, Star, Triumphator etc.

14.4 Calculators with a Built-in Multiplication Table

14.4.1 Barbour

The first attempt to set Napier's scheme (of his bones) to mechanism that would add and register the overlapping ordinal values was patented by E.D. Barbour in 1872. His accumulator mechanism, including the numeral wheels and their devices for transferring the tens, is mounted in a sliding carriage at the top of a machine, which may be operated by a hand-knob. Extending through the bottom of the carriage is a series of pinions, one for each ordinal numeral wheel, and connected thereto by a ratchet and pawl action. The pinions are each so arranged as to be operative with a gear rack beneath the carriage when the carriage is slid back and forth. Thus the wheels received action from one direction of the motion of the carriage and remain idle during the movement in the other direction.

14.4.2 Ramon Verea

It was in 1878 when Ramon Verea, a Spaniard living in New York invented the first direct multiplying machine: by one turn of the crank for each figure in the multiplier rather than the previously used repeated additions. Verea asserted that he did not make the machine to sell the patent or to put it to use, but simply to show that it could be done and that a Spaniard could invent as well as an American.

14.4.3 Léon Bollée [4)]

Figure 14.25

Lé on Bollé e (1870 - 1913) from Le Mans, France, constructed in the course of three months (from February until April 1888) a calculating machine in which the multiplication table was mechanically represented. Bollé e's father had a factory for constructing steam tramways and later turned to the manufacture of automobiles. Lé on's machine is especially remarkable because the designer was not yet eighteen years old when he built his machine and knew no details of other calculating machines. His invention was designed primarily for multiplication and division, and contained a series of what Lé on called « controllers» , which may be described as material embodiments of the multiplication table, see Figure 14.25. Instead of using eighty-one multiplying gear racks for each order as in the

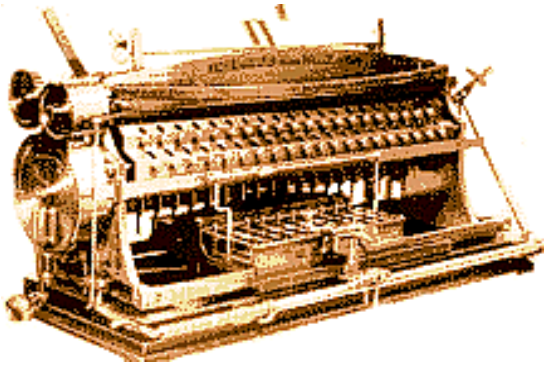


Figure 14.26

Barbour patent, Bollé e used but two gear racks for each order: one for adding the units and the other for adding the tens; these racks operated vertically. The racks are frictionally held against gravity in the permanent framework of the machine, and are moved up and down by contact at each end, received from above by a bar and from below by the pins of varying length stored in the movable plate, see Figure 14.26.

The length of each rod indicated the result of a multiplication by the digit associated with its base. For example the mechanism for 5 would have a sequence of rods of lengths 5, 10, 15, 20,…, two digits being represented by two rods of the appropriate length. By employing ten multiplication bodies side by side in the rear of the setting mechanism and adjusting them individually in one direction to represent the figures of the multiplicand, and adjusting them as a unit in another direction to represent successively the figures of the multiplier, the product could be obtained by operating the transmitter as many times as there are figures in the multiplier. When a multiplication was being performed, the mechanism would add to the result register an amount equal to the lengths of the rods in use.

The accumulator mechanism is mounted in a reciprocating frame which moves horizontally, causing the gears of the numeral wheels to engage first the units racks on their upstroke under action of the pins, and then the tens racks on their downstroke under the action of the top bar of the vertically moving frame, the downward motion, of course, being regulated by the upward movement it receives from the pin that forces it up.

Although the Bollé e machine, see Figure 14.26, is reputed to be a practical machine, as it is attested from the models on exhibit in the Museum of Des Arts and Metiers of Paris, it was never manufactured and placed on the market; it was typically quite expensive and large and cumbersome in use. Lé on focused his attention to the construction of racing cars.

14.4.4 The Millionaire [6)]

In 1892, **Otto Steiger** (1858 - 1923) patented a casted one-piece *built-in multiplication table* and also started manufacturing a multiplication machine based in Verea's and Bollé e's approach. This machine was manufactured between 1895 and 1935 by Hans W. Egli of Switzerland and sold with the brand name *The Millionaire*, see Figure 14.27. This machine is to be regarded as a proper multiplication machine in that it solves problems of multiplication directly on the basis of the multiplication table, whereas other types of calculating machines are adding machines and, as such, carry out multiplication by a continued series of additions. The main advantage of the Millionaire, as again all other types of calculation machines, is the astounding speed with which it operates, especially while doing multiplication and division. Each place of the multiplier or quotient requires only a single turn of the crank **C**, during which the necessary displacement of the result occurs automatically.

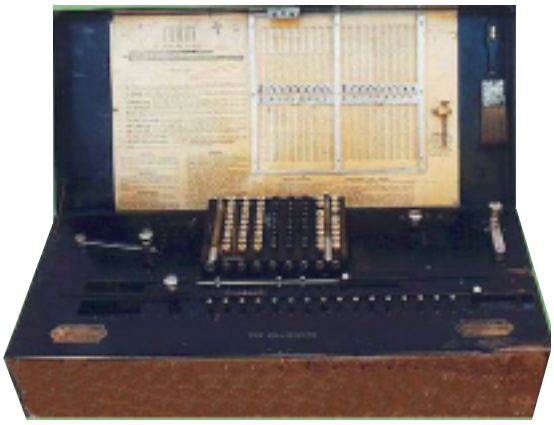


Figure 14.27

The multiplication mechanism consists of one horizontally so-called multiplication-table-body

T and its setting device that permits it to move in vertically direction by the lever **L** on the left.

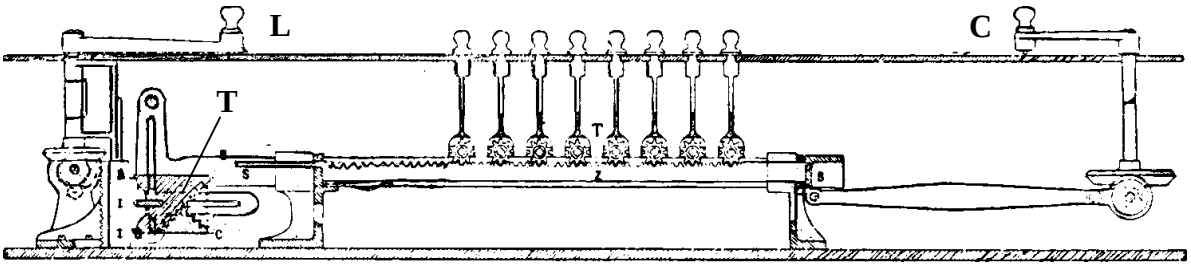


Figure 14.28 Longitudinal section of the Millionaire

The transfer mechanism consists of nine parallel racks and the shafts positioned transversely across these racks. Along these shafts the setting wheels may be shifted in the customary manner by means of the knobs on the setting board (Figure 14.27 is a Millionaire with key-board operation of later production) and this can be brought into mesh with any of the nine racks, depending on the value of the respective place of the multiplicand, see Figure 14.29.

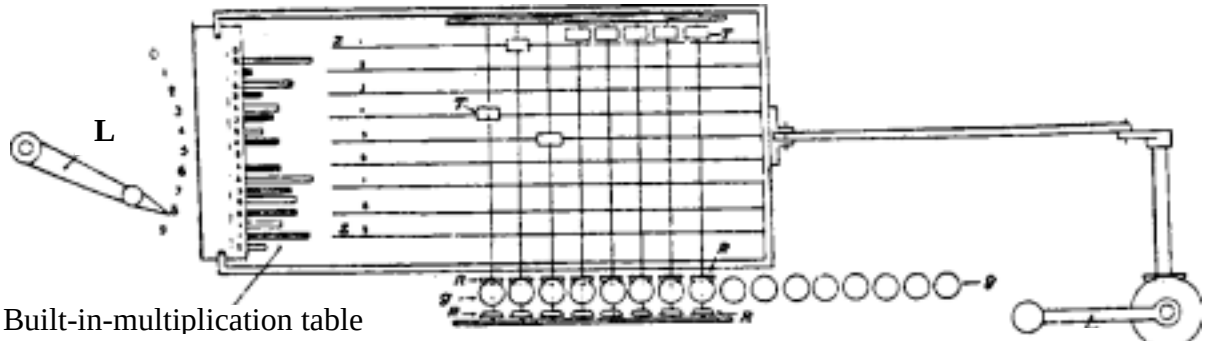


Figure 14.29 Interior view from above **C**

On each of these shafts a pair of bevel gears is also mounted, which are shiftable in their axial direction. The bevel gears transmit the rotation of the drive gears, which corresponds to the longitudinal movement of the racks onto the register mechanism in a positive sense in case of multiplication and in a negative sense in case of a division. Appropriate shifting mechanisms move these bevel gears periodically into and out of mesh with the register mechanism so that only the forward movement of the racks is transmitted while the return movement of the racks has no effect upon the register mechanism. The ends of the racks are either opposite the tens group or the units group of the teeth of the multiplication table. The alternation of the groups is effected by a small, horizontal, transverse shift of the multiplication table, which occurs automatically. Example of a multiplication: 434294 x 7. First set the reversing lever on X; then enter the multiplicand 434294 by shifting the setting slides (or by pushing the keys); then enter the multiplier by shifting the lever **L** to the 7 on the scale, with the result that the multiplication table *x* x 7 is placed opposite the ends of the racks.

During the revolution of **C** of 360^0, the racks are pushed twice against the multiplication table.

0 0 4 3 4 2 9 4

0/0	0/0	2/8	2/1	2/8	1/4	6/3	2/8

7

$0 – 90^0$: the racks are pushed forward and an amount corresponding the tens = 00222162 is set in the register: → 02221620

$90^0 – 100^0$ idle return of the racks, displacement of register 1 place to the left 00818438

$180^0 – 270^0$ transversal displacement of the multiplication table 02***0***39***0***58

and transmission of the units values and preparation for carrying tens 03040058

$270^0 – 360^0$ idle return of the racks and transfer of the tens carried from the previous addition.

Some models weighed up to 60 kg. About 4655 units were sold.

14.5 The Proportional Gear Wheel Mechanism

Rodney and Alfred Marchant, 1910 [5)]

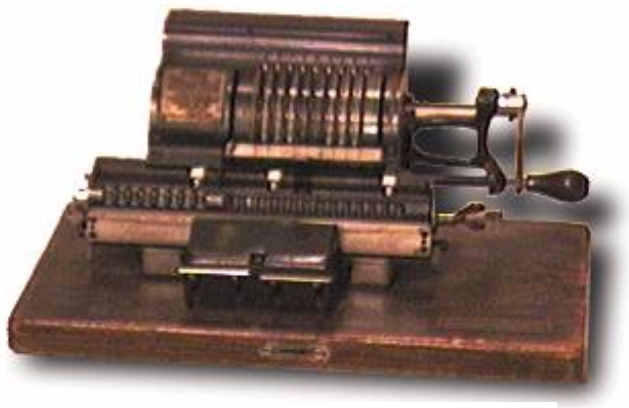


Figure 14.30 Marchant

The Marchant brothers Rodney and Alfred founded in 1910 their calculator factory in Oakland.
They provided the machine with the same exterior design as that of the Odhner machines and also copied its dimensions, see Figure 14.30.

Of all calculating machine systems so far described the result wheels are rotating during a different time, dependent of the figure that has to be added in the register system. For example, adding 19 in the register, the result wheel, which adds the 9, will rotate 9 times longer than the other one. The effect is a by fits and starts operating register system that limited its speed. The advantage of the proportional gear wheel system is the equal rotation time of every result wheel independent of the figure to be added. By this non-interrupted movement of the result wheels and the tens carry the speed can be raised up to 1300 hits per minute.

The principle of this proportional system is based on a complete 9-step wheel-work for every digit in the setting register, see Figure 14.31.

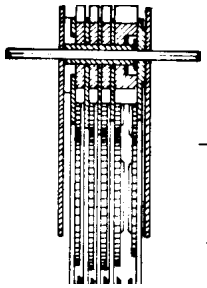
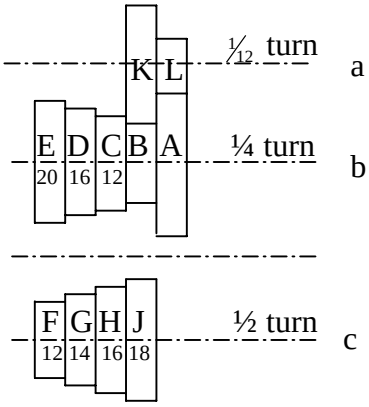


Figure 14.31



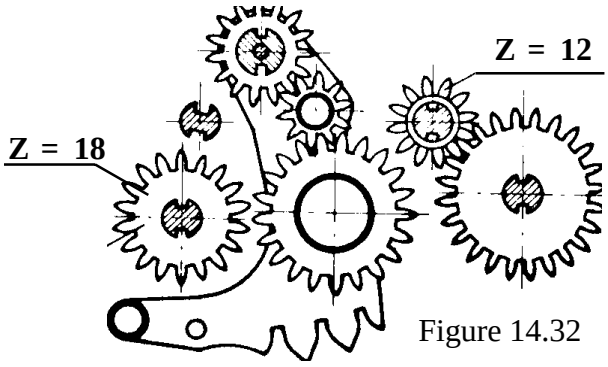


Figure 14.32

During the course of a calculation operation no part of the wheel-work is out of contact and so the result wheels will always rotate correctly. Every wheel-work consists of 5 swing-sheets, see Figure 14. 31, with 11 different gear wheels, which can swing into two positions, any of them make possible two different transfers.

The first set of gear wheels with a number Z of 18, 16, 14, and 12 teeth rotate 180^0 = ½ per full turn of the crank so that 9, 8, 7, or 6 teeth per full turn are revolved.
Wheels with 20, 16, or 12 teeth rotate 90^0 = ¼ per full turn of the crank so that 5, 4, or 3 teeth per full turn are revolved.

Wheels with 24 teeth (K) and 12 teeth (L) rotate via an interjacent wheel with 30^0 = $1/12$ per full turn of the crank so that 2 teeth or 1 tooth per full turn are revolved.
The wheels indicated by z = 12 and z = 18 in Figure 14.32 are part of the 9-step wheel-work.

14.6 The Proportional Levers The Mercedes-Euklid [6,7)]

In 1903 the proportional lever used as guidance element for calculating machines was developed by the German engineer **Christian Hamann** (1870 - 1948) at Berlin. From 1905 these proportional lever machines were produced by Mercedes Office machine manufactory in Berlin-Charlottenburg.

Figure 14.33

The principle upon which all Mercedes-Euklid models have been based differs entirely from the calculating machine systems so far described. Innovations and improvements are: 1) Automatic division: contrary to the usual method of division, the Mercedes-Euklid has completely automatic division, which is carried out entirely mechanically by rotation of the crank and manipulation of the reversing buttons; and 2) automatic carriage movement. Figure 14.33 shows model 1, 1905, a manual machine with slide setting.

The actuating mechanism consists of ten parallel racks with a pin on one end which left through a slot of the proportional lever D –D' and the square shafts positioned transversely across the racks. Along these shafts the setting wheels may be shifted in the customary manner by means of the knobs on the setting board. Each of these shafts is connected to the register mechanism. The forward movement of the racks is transmitted while the return movement of the racks has no effect upon the register mechanism.

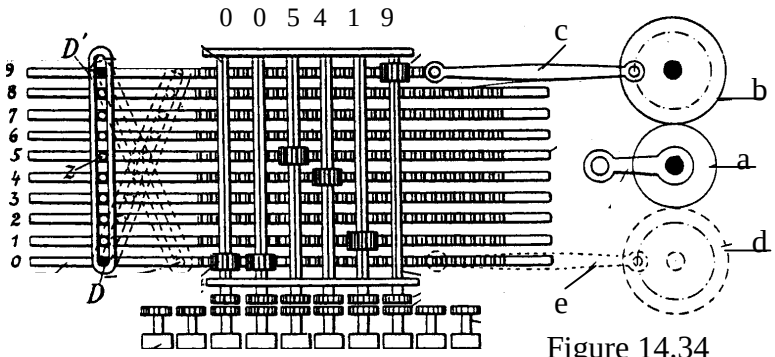


Figure 14.34

The racks are proportionally displaced by the lever D – D'. With the reversing button by which the mechanism may be set for ADD.MULT. (addition / multiplication), wheel **b** and connecting rod **c** let proportional lever turning about D; or set for SUBT.DIV. (subtraction / division), wheel **d** and connection rod **e** let proportional lever turning about D'.

14.7 Control-catch mechanism [5,8)] The Hamann-Manus, 1927

The designer **Christian Hamann** of Berlin in 1925 constructed a calculating machine with control-catch mechanism. Hamann had provided the machine with the same exterior design as that of the Odhner machines and also copied its dimensions, see Figure 14.35.

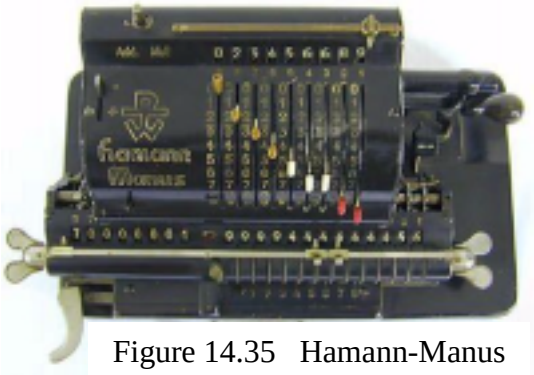


Figure 14.35 Hamann-Manus

The interior design of the machine, however, is completely new and original. Its most important working part is not the pinwheel itself but a geared wheel with a thirty-six-part inner and outer gearing, see Figure 14.37. It should be stressed that the Hamann-Manus is the first and only small, manual calculating machine with completely automatic division. This sets it apart from all similar calculating machines.

In figure 14.36 part **2** (fixed to the chassis) and part **3** are two disks next to each other with over a part of the circumference a smaller diameter; the opening as shown in the figure can be changed from maximum to closed by revolving part **3** with the lever **4**. Fixed to the shaft **1** is part **6** with attached to it part**7**, the *control catch.* Pushed by a spring (not shown here) roller **8** is pressed against the disks **2** and **3**. If shaft **1** revolves, roller **8** can enter the opening between part **2** and **3** that causes a connection between catch **7** and part **5**, and so between shaft **1** and part **5** until roller **8** leaves the opening. In short: lever **4** does not participate in the revolution of shaft **1**; for every 360^0 revolution of shaft **1**, wheel **5** only rotates as many teeth as are set by lever 4.

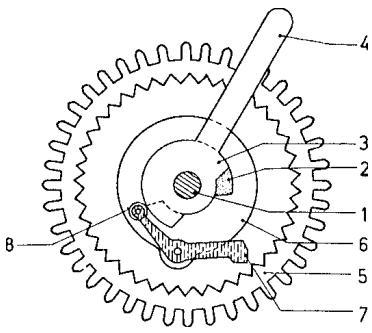


Figure 14.36

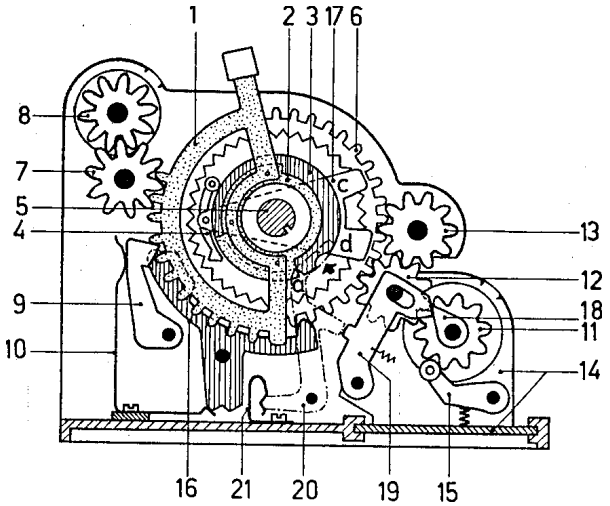


Figure 14.37

In the transverse section Figure 14.37, lever **1** is set on zero, the opening between the two disks **2** and **3** is closed, wheel **6** will not rotate when shaft **5** revolves. Part **1** is provided with teeth which have 2 functions. First, part **7** and part **8** will rotate if **1** is set; part **8**, a number wheel, indicates the number to be add in the result mechanism or indicates the multiplicand; second, part **1** is locked by part **9**. Part **11,** the result wheel, either clockwise turned for addition and multiplication, or counter clockwise for subtraction and division with the help of an intermediary gear wheel (The wheels **11**, **12**, and **13** are not in the same plane). The carry mechanism is activated by tooth **18** of part **11**, which moves part **19** to the left, on its turn part **19** moves part **20**. Inactivated part **20** covers an opening, see arrow ; activated the opening let the roller moves wheel **6** one position.

The advantage of this system compared to the variable teeth system especially is the possibility to use the from 1900 through going cheap punch technique in stead of the more expensive plane and mill technique.

1) Michael R. Williams A History of Computing Technology

2) Friedrich W. Kistermann IEEE Annals of the History of Computing Volume 21 1999

3) Edgar P. Vorndran Entwicklungsgeschichte des Computers

4) F. Trinks Geschichtliche Daten aus der Entwicklung der Rechenmaschine

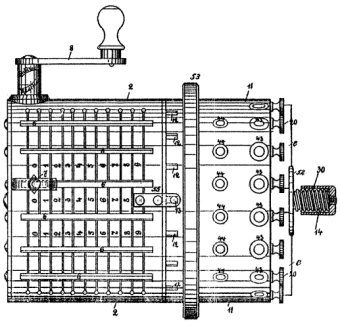
5) Martin Graef 350 Jahre Rechenmaschinen / The calculating Machines

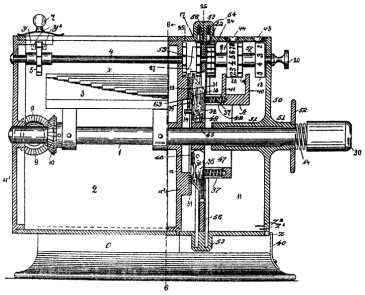
6) A. Galle Mathematische Instrumente

7) K. Lenz Die Rechenmaschinen und das Maschinenrechnen

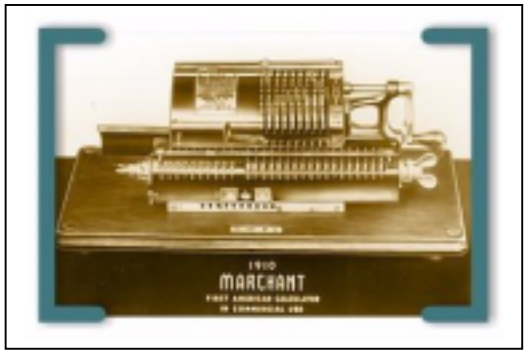
8) A. Rohrberg Theorie und Praxis der Rechenmaschinen

Besides this calculating machine, and the machines of § 14.2.1.3, and § 14.6.1, Christian Hamann was the designer of a forerunner of the CURTA, which he patented in 1902.





The drawing may speak for them selves.



Chapter fifteen, *Tally sticks*

The word tally – kerfstok in Dutch – still lives on in the Dutch saying: "Iets op zijn kerfstok hebben" = "Having something on his record", with the meaning of having done some wrong. A historical example is a quotation of Carel van Mander, a contemporary of the painter Pieter Brueghel de Oude (†1569) who wrote about him: "While he was in Antwerp, he lived with a young girl whom he would have married if she had not had the unfortunate habit of constantly telling lies". *Hij maeckte met haer een verbondt, hy soude al haer loghenen kerven op eenen kerfstock en so den kerfstock met der tijdt quam vol te worden, soude 't Houwlijck gantsch uyt en te nieten zyn, ghelijck het eer langhen tijdt gheschiede.* (He made a pact with her, that for each lie she told he would cut a notch in a tally, and when the tally was filled up with notches, the marriage would be dropped and given up, as it happened after time.).
In this case the tally was used for recording a number. There are many examples of the use of notched sticks in different parts of the world. Traces of this type of recording system can be found from as early as the Upper Palaeolithic cultures of 30,000 B.C.
Of the saved tallies, the oldest all of bone, the Ishango bone is a good example.

15.1 The Ishango bone

By the shore of Lake Edward on the borders of Uganda and Z aire, about 20,000 years ago lived near the present-day Ishango, a small Neolithic community. The settlement had a relatively short life span of a few hundred years before being buried in a volcanic eruption. One of the unearthed objects there is a bone tool handle which has a little sharp piece of quartz firmly fixed in a groove at one end, now at the Musé e d'Histoire Naturelle at Brussels, see Figure 15.1.

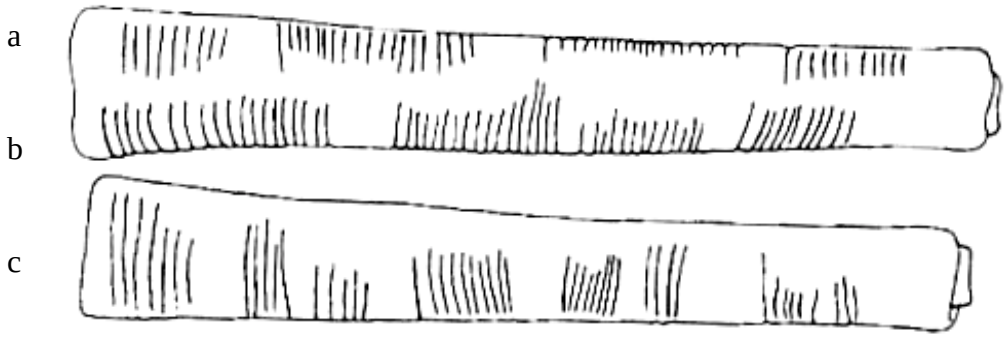


Figure 15.1

The remarkable fact of this Ishango Bone, as it is called, is a number of notches arranged in three distinct rows:
- The first row (a) contains 4 groups of notches, respectively 9 19 21 and 11 (sum = 60) or 10 - 1 20 - 1 20 + 1 and 10 + 1;
- the second row (b) also contains 4 groups, respectively 19 17 13 and 11 (sum = 60) the 4 prime numbers between 10 and 20 and
- the third row (c) continues with two prime numbers following on 19, 17, 13 and 11: namely 7 and 5; then 5 – 10, 8 – 4 and 6 – 3, the concept of duplication or multiplying by 2.

By the organisation of the notches it is not plausible that it happened by accident. For the possibly function one only can guess.

15.2 A Finnish Tally stick

After the Roman era the knowledge about the Roman figures in entire Europe only was known in monasteries to monks and the writers serving courts and cities, who were writing on parchment. Paper was produced in Europe from the 14th century and was expensive and writing a difficult job. The rest of the population ‘recorded’ their data about quantity cattle, about harvest, payments, demands and debts on tallies.
An other saying was: "Buy upon stick" with the meaning of buy on credit. For that purpose tallies were used, which were split lengthwise. The two parts fit to one other on an unique way. On both parts the name or an identification mark was recorded. At every entering or writing, the two parts were put against each other and carved together. One part stayed at the shopkeeper or creditor, the customer took the other part home. Commiting fraud was not possible. Would the customer write off a part at home, than the deceit was brought to light if the two parts were put against each other at the shopkeeper’s. Nor could the shopkeeper enter anything unnoticed. Such a tally from Finland is shown in Figure 15.2.

Figure 15.2 Finnish tally stick to record units of work done (25 x 2 x 1,5 cm)

15.3 The French Code Civil and the Dutch Burgerlij k Wetboek (Civil Code)

The Lords and Kings of France used to raise taxes, named ‘taille’ = carve (from the Latin taliare), from their serfs and citizens, which simply beared this name because the tax-gatherers had the custom to indicate with a carve in a tally what every one had to pay. This way of ‘keeping the books’ can be found in the Code Civil:

In the French Code Civil from 1804 we read in article 1333:
Les tailles corrélatives à leurs échantillons sont soi entre les personnes qui sont dans l’usage de constater ainsi les fourniture qu’elles sont ou reçoivent en detail:

"Tallies, corresponding with their double, deserve trust between those who use to proof on such a manner the deliveries, which they do or receive by retail".

In The Netherlands, from 1811 the French Code Civil was in effect. After Napoleon in 1838/1842 the Dutch Burgerlijk Wetboek (B.W.) was introduced, largely a translation of the Code Civil. This B.W. is several times modified. In the edition of 1988 (not a misprint!) the means of proving in civil matters reads, article 1902 to 1982, among others: article 1924:

> **Kerfstokken**
>
> **Art. 1924** Kerfstokken, met hun dubbel overeenkomende, verdienen geloof tusschen degenen die gewoon zijn om de leveranciën, welke zij in het klein doen of ontvangen, op dusdanige manier te bewijzen. (B1904, 1911)
>
> (1904 concerning written proof; 1911 concerning by agreement)
>
> Art. 1924 Tallies, corresponding with their double, deserve trust between those who use to proof on such a manner the deliveries, which they do or receive by retail.

15.4 Exchequer tallies

In no land tallies have been used so long and so official as the Exchequer tallies in England. From the *Dialogus de Scaccario* from 1186 of the Exchequer **(see page 48)** is known what the standardised notches, *the incisione talearum*, looked like and which values were indicated by these notches**.** See Figure 15.3 showing original tallies from the 13th century.

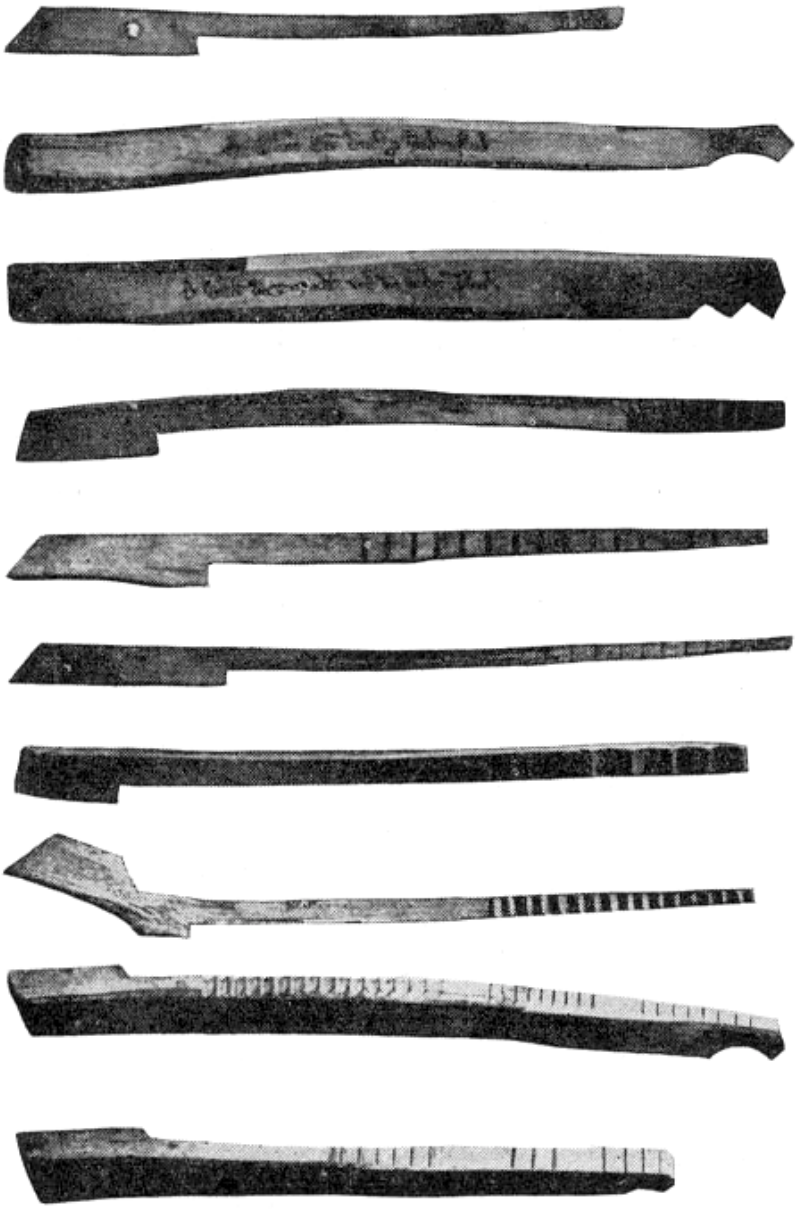


Figure 15.3

The notch for £1000 (librae) is placed at the end and is as large as the hand is wide (see uppermost tally), cut a straight part from the end;
for £100 the notch is as large as the thickness of a thumb, and to distinguish it from that for £1000 it is not straight but curved (see second tally);
for £20 (a score) it is as large as the thickness of the little finger (see third tally);
for £1 it has the breadth of a ripe barleycorn (see fourth tally);
for 1 shilling it issmaller but still large enough to be seen as a notch (see fifth tally), and
for one penny a cut is made, with no wood being removed (see sixth tally).

Till about the end of the 18th century the amounts about which consensus was reached between the rate payer and the civil servant were recorded by ‘cutting tally sticks’.

The tally was made of a piece of elm-wood, split in such a way that the notches would appear on both halves, therefore producing two copies, or records, of a transaction. At some time concerning amounts were recorded also in books, but the payment receipts remained in the form of the ‘foil’ (folio = leaf), the thinner part of the tally. The ‘Treasury’ also kept the accompanying ‘stock’. This system was abolished with the law of 1783, which mean that no new tallies were emitted. From then a paper cheque was handed over. The tallies still deposited with the Treasury remained valid and therefore were kept to 1826.

In 1834 it was found that there was a considerable accumulation of tallies at the treasury; and the question then arose, what was to be done with such amount of old bits of wood? Nothing could be easier than to allow them to be carried away for firewood by the miserable people who lived in that neighbourhood. However, they never had been useful, and official routine required that they should never be, and so the order went out that they were to be privately and confidentially burned. It came to pass that they were burned in the stove in the House of Lords. The stove, over-gorged with these preposterous sticks, set fire to the panelling; the panelling set fire to the House of Commons; the two houses were reduced to ashes!
The painter Turner used this fire as subject of two of his paintings, see Figure 15.4, the burning of the Houses of Lords and Commons, 16th October 1834 by Turner, who witnessed the burning from a boat on the River Thames.

Figure 15.4

When a person deposited money in the Bank of England the amount of the deposit was recorded on a tally, the tally was split, and the bank kept the *leaf* (foil) and the depositor kept the *stock*. The depositor thus became a *stockholder* in the bank!
One does not have to guess after the origin of the word *stock exchange*.
Also the word *tallyman* and a *tallyshop* now is explained.

Chapter sixteen, *Finger numbers, Finger reckoning*

Finger numbers is a system of representing numbers by use of the fingers - *digits* - in Latin. In Medieval Latin the word *articuli* was used to indicate pure tens or hundreds, referring to the joints of the fingers employed in representation of tens. Gradually this representing of numbers was 'international' standardised. Number representation on the fingers was eventually spread throughout all of North Africa, the Middle East, Europe, and might have stretched into India and parts of the Far East during many centuries, for there are numerous references to it in Greek, Roman, Hebrew and Arabic literature.

16.1 Ancient testimonies about finger numbers

The Greek poet Aristophanes (ca. 445 to 385 B.C.) wrote in his *The Wasps*: "First one have to calculate the earnings of the city of Athens, then the cost of the payments, not laborious with pebbles, but at once with the fingers."
Juvenal (62 to 142 A.D.) refers to the custom to represent numbers below 100 on the left hand and the hundreds on the right in his tenth satire, saying: "Happy is he indeed who has postponed the hour of his death so long and finally numbers his years upon his right hand."
That the system was familiar to the people is evident from a remark of Pliny (23 to 79 A.D.) to the effect that King Numa (6th century B.C.) dedicated a statue of two-faced Janus, the fingers being put in a position to indicate the numbers of days in a common year.

16.2 A recent use of finger numbers

The representation of numbers with fingers as a kind of sign language was required at (barter)trade to communicate numbers across language barriers or to overcome the noise in the marketplace. This was in use around the whole world.
A photograph, taken in 1956 in Southern India [1)], see Figure 16.1, shows a pearl merchant and a buyer who are negotiating for the price.
This happened without words, and hidden from other observers by using finger numbers with the hands under a cloth, hiding the price-making process from competitors.

Figure 16.1

16.3 Tesserae, Roman game discs with finger numbers

These Roman tesserae from the first century A.D., made of bone or ivory are found in great numbers with clearly indicated finger numbers, but only from 1 to 15. Therefore it is supposed that they are playing disks. Figure 16.2 shows two disks of ivory showing 8 and 9 formed by finger gestures. These values are also in Roman figures VIII and VIIII on the backside.

Figure 16.2

16.4 Counting on the fingers

If several numbers had to be added together, the first of them was noted on the fingers, the second added to it mentally and the result noted on the fingers, the third added mentally to the total noted on the fingers, and so on. If one was interrupted, it was only necessary to hold the hands in their current position and the calculation could be taken up again later without the bothersome detail of attempting to remember the last result.
In a sermon of St. Augustine, (354 – 430 A.D.), as bishop of Hippo in North Africa, he cited from the evangelic of John, verse 21,11 where Peter by order of Jesus goes out fishing and catches 153 fishes. Augustine gave a mystically meaning to this number: add all numbers from 1 to 17 (10 and 7 are special numbers; 10 is the number of the 10 commandments, 7 of the Holy Ghost). Than he said: "Count again and reckon 1 + 2 is 3, 1 + 2 + 3 is 6, 6 + 4 is 10 and 10 + 5 is 15 etc. Finally you attain to 17 and you bear 153 with the fingers":
Sic pervers usque ad decem et septem, portans in digits centum quinquaginta tres.

16.5 The oldest description of finger numbers

The oldest complete description of finger numbers is of Bede, 8th century (see § 5.1).
In 725 A.D., Bede wrote a treatise, *De Temporum Ratione*, on the calendar together with information for calculating the date of Easter. This subject required his readers to be familiar with methods of calculation, so to this he devoted an introduction, *De Computo vel Loquela Digitorum*: to the computing and language of the fingers. In this he gave a complete explanation about the position of the fingers on counting.

This description of the position of the fingers is based on the Latin description of Bede, not a description of the various pictures, in which are several errors.
For denomination the fingers, read *l* for the little finger, *r* for the ring finger, *m* for the middle finger, *f* for the forefinger and *t* for the thumb.

For representing 1 to 6 by the left hand and 1000 to 6000 by the right hand, the fingertips of *l*, *r* and *m* are bent at the middle joint;
for representing 7 to 9 by the left hand and 7000 to 9000 by the right hand, the *l*, *r* and *m* are closed on the palm, as represented by the tesserae of Figure 16.2;
for representing the tens 10 to 90 the *f* and *t* of the left hand are used;
for representing the hundreds 100 to 900 the *f* and *t* of the right hand are used, see Figure 16.3 [2)],
a picture from *De Numeris* of Noviomagus from 1544.
The numbers were expressed by the hands being held upward, the palm of the hands flat to the observer. Now we see the following system:

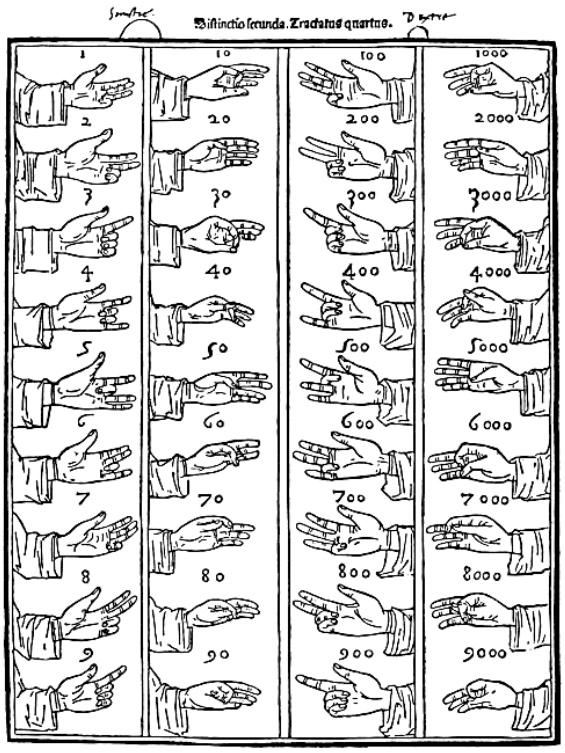
left hand		right hand	
l r m	*f t*	*t f*	*m r l*
units	tens	hundreds	thousands
1 – 9	10 – 90	100 – 900	1000 – 9000

A place value system!

Figure 16.3
From *De Numeris* of Noviomagus

Bede described: *Quum ergo dicis unum, minimum in laeva digitum inflectens, in medium palmae artum infiges,* which means:

If you say one, than you have to bend the l of your left hand and lay its tip on the palm,
and he continues:
for two, lay down *r* next to it;
for three, also *m*;
for four, stretch the *l* again;
for five, also stretch the *r* again;
for six, stretch the *m*, but lay down the *r* again;
for seven, stretch all the fingers and lay down only *l* closed to the palm;
for eight, lay also the *r* closed to the palm and
for nine, also the *m.*
(For eight and nine see the tesserae, Figure 16.2)
Up till now only three fingers of the left hand are used for the units, the *digits.*
For the tens the *f* and *t* are used:
for 10, tip of *f* touching middle joint of *t;*
for 20, tip of *t* touching base between *f* and *m;*
for 30, tips of t and *f* forms a circle;
for 40, *t* and *f* vertical and close together;
for 50, *t* bent at both joints, flat against the palm;
for 60, tip of *f* bent over *t* of 50;
for 70, 1st joint of *f* held on 1st joint of vertical *t*;
for 80, tip of *f* held on 1st join of *t* and
for 90, *t* bent over the 1st join of *f*.



left hand right hand
from the *Summa de Arithmetica* of Luca Pacioli [1)]
Luca has changed 100 and 1000
Figure 16.4

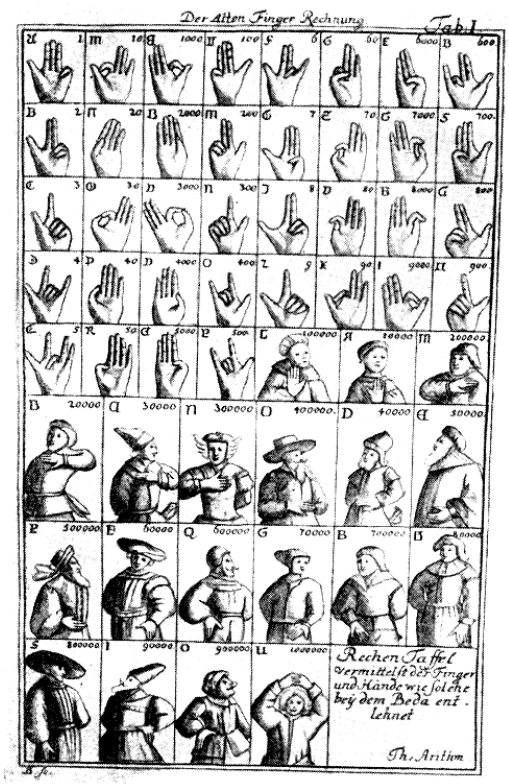


Figure 16.5

The first secular - not clerical - book, at the same time the first printed in which finger numbers were used is: *Summa de Arithmetica, Geometrica, Proportioni et Proportionalita* of Luca Pacioli, published in 1494. See Figure 16.4.

Leonardo of Pisa (1170 - 1240), see § 8.15, also wrote: "When the mentioned figures of the Hindus and the place value system by exercises are well be mastered than it is very useful for them who wish to become skilful and competent in arithmetic to learn reckoning with finger numbers as formerly the ciphering masters that considered to be very sensible".

As late as 1727 Jacob Leupold discussed the finger numbers of Bede in his *Theatrum Arithmetico – Geometricum,* see Figure 16.5 [1)]

Arab finger numbers are the same as Bede's, but right and left are changed, the units start right as their writing runs from the right to the left.

Counting on the fingers stayed popular in the West to about 400 years ago: arithmetic handbooks were considered not to be complete if the method not was incorporated: see *Theatrum Arithmetico – Geometricum* of Jacob Leupold from 1727! with the finger numbers of Bede. However these are increased with numbers from 10,000 tot 100,000.

16.6 Samuel Pepys

For us is hard to imagine that multiplication of small numbers was a difficult process for an educated person, but even as late as the 1600s the subject of multiplication was often left until university level studies. Samuel Pepys (1633 - 1703), - an often-quoted person by his exact kept up diaries - was well educated by the standards of his day. In 1662 he got, more by his descent than by his knowledge, an appointment as Secretary to the Admiralty. Then he was forced to hire a private tutor to teach him the multiplication and division skills for the job. By those diaries we know how difficult that was.

16.7 Finger computation

Completely different from the scheme described above, it was also possible to perform multiplications using the fingers.

Example 1, (for numbers less than ten), multiply 7 by 8, raise two fingers on one hand (since 7 - 5 = 2) and three on the other (since 8 - 5 = 3). Multiply the total raised fingers: (2 + 3) = 5 by 10 = 50, add the product of the others (including the thumbs), 2 x 3 = 6, (the former result is the tens, 50, and the latter is the units, 6), the result being 56.

Example 2: multiply 9 by 6: raise on one hand (9 - 5) = 4(a) fingers and on the other (6 - 5) = 1(b) finger. Multiply the total raised fingers: (4 + 1) = 5 by 10 = 50, add the product of the others: 1 x 4 = 4, result = 50 + 4 = 54. You will have to carry, as the units are more than ten. This depends, of course, upon the fact that (10 - a)(10 - b) = 10[(5 – a) + (5 - b)] + ab.

Example 3, (for numbers from 10 to 15). One has to know that the square of 10 equals 100.
12 x 13: bend on one hand (12 - 10) = 2 fingers and on the other hand (13 - 10) = 3 fingers.
Multiply the total bent fingers: (2 + 3) = 5 by 10 = 50;
add 2 x 3 = 6, and 10 x 10 = 100, result 50 + 6 + 100 = 156.
This method is evidently general, since (10 + a)(10 + b) = 100 + 10(a + b) + ab.

Example 4, (for numbers from 15 to 20). One has to know that the square of 15 equals 225. 18 x 16: bend on one hand (18-15) = 3(a) fingers and on the other hand (16-15) = 1(b) fingers.
Multiply the total bent fingers: (3 + 1) = 4 by 15 = 60;
add 3 x 1 = 3, and 15 x 15 = 225, result 60 + 3 + 225 = 288.
This method is evidently general, since (15 + a)(15 + b) = 225 + 15(a + b) + ab.

Using this method it was unnecessary to learn the multiplication table above 5 x 5.

1) Karl Menninger
Number Words and Number Symbols A Cultural History of Numbers

2) Louis Charles Karpinski
The History of Arithmetic

Chapter seventeen, *China*

Chi Hwang-ti, the first emperor of the Tsin dynasty, ordered all books to be burned in 213 B.C. There are some texts which survived the purge, dating from about 530 B.C., which indicate that the Chinese had a well-developed decimal place value notation that had been in use for several hundred years. Thus this system had to be older.

The Chinese language only knows words existing of one syllable. Each word has his own symbol e.g. 两, 俣 or 刼. This also is in effect for numerals. In other words, each figure or number has its own symbol, a condition for a place-value system, but by that also an aid to get the idea for the place-value system

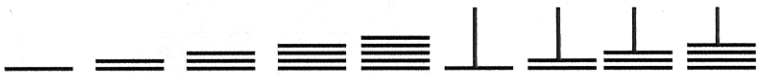
Before 300 B.C., numbers, also the tables of multiplication, were written with those symbols. Calculations were made, if not to do by heart, with counting-rods. After that or at the same time numbers were represented by copying counting-rods. Compare this to calculation with counters 'on the lines' in medieval Europe and make notes of the result in Roman numerals.

17.1 Chou, Chinese counting-rods and rod-numerals

It is certain that in China in the 6th century B.C. already calculations were made with counting-rods. A complete set of rods contained 271 pieces, each of which was about 6 mm in diameter and ca. 40 to 80 mm long. They were constructed from bamboo, wood, bone, horn, ivory, iron and sometimes even jade. The rods came in two colours, red for use when representing positive quantities and black for negative ones.
The integers from 1 to 5 were just represented by simple vertical rods but, for representing numbers from 6 to 9, a horizontal rod representing five together with enough vertical rods to make up the required digit. For the tens the numbers were rotated 90 degrees. As the units the rods for hundreds, ten thousands, or millions were vertical, tens, thousands, or hundred thousands horizontal. A centesimal system:

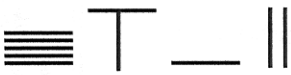
1 2 3 4 5 6 7 8 9 and
100 to 900 etc. and

heng rods for units, hundreds etc.



10 20 30 40 50 60 70 80 90 and
1000 to 9000 etc.

tsung rods for tens, thousands etc.

Example of a number: 5612 = 

To avoid mistakes on account of the lack of a any device to indicate zero, one calculated on a large squared board, with each digit of a number occupying a separate square, see Figure 17.1
It is known that for a calendar reform done in the year 1281 A.D. the calculations were made on such a board. At that time numbers including a zero were written with a square in place of the zero:
example of a number including a zero: 4036 written as: ≡ □ ≡ ⊤ or as : ≡ ○ ≡ ⊤
These numbers written like rods are called San-ju numbers.

17.2 The calculating board for rods

On this squared board the numbers were placed horizontally, one number in each square, the lowest value at the right. The arithmetic board of Figure 17.1 is a Japanese equivalent of the Chinese board on which a Japanese is performing a calculation with the counting-rods by which these rods do not lie on the board according the above explanation.

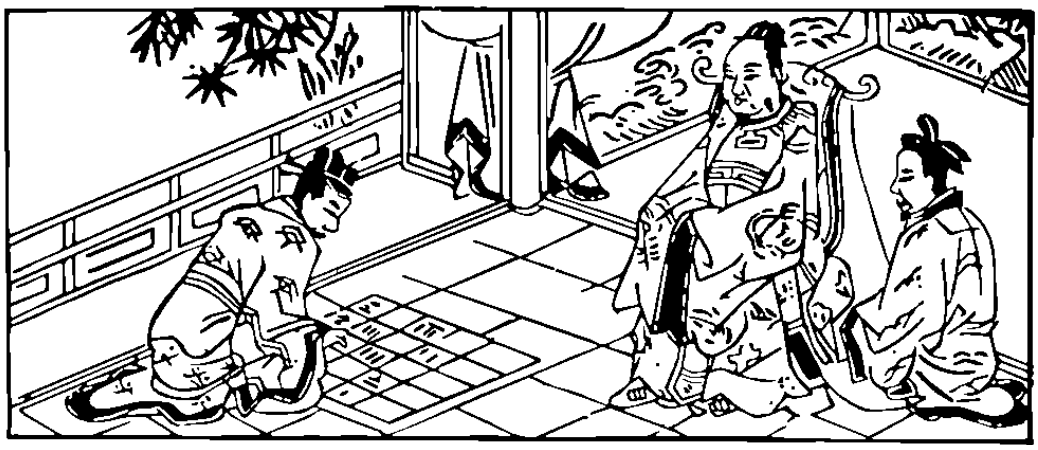


Figure 17.1 Calculating on a (Japanese) counting-rod abacus

On this abacus with rods one could perform addition and subtraction and also multiplication, division and even extracting roots. This calculating is to compare to calculating with counters ‘on the lines’ by which a trained calculator could calculate so fast that one described it in that time as "the hands were flying like swallows over the board" **.**

17.3 A ‘Pascal’s’ triangle

On this arithmetical triangle, later known by the name ‘Pascal’s triangle’, San-ju signs are used for the numbers.
It is to see clear that for 10 ⊖ and for 20 ⊜ is used. Sometimes horizontal and vertical were interchanged: see Figure 17.2, the front page of the Ssu Yuan Yii Chien by Chu Shih-Chieh from 1303 A.D.

The rods were used till ca. 1600 A.D. From ca. 1300 A.D. they got competition of the suan-pan with beads.
It is known that at another calendar reform of 1384 A.D. this suan-pan was used.

To understand the calculations on a suan-pan of the next examples it should be helpful to perform them on a suan-pan.

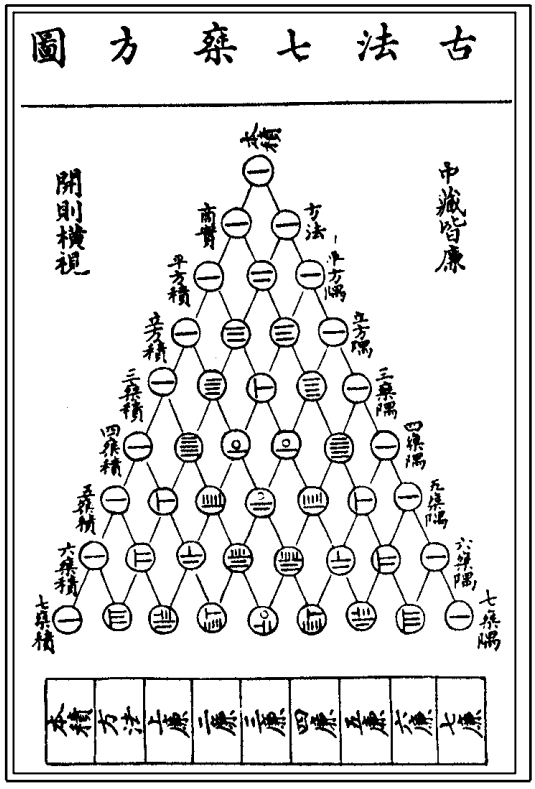


Figure 17.2

17.4 SUAN-PAN or calculation by means of beads

There is a sixth-century Chinese reference[1)] to an abacus on which counters were rolled in grooves. The description of this ancient Chinese abacus and the known intercourse between East and West give us a good reason to believe that the Chinese abacus was suggested by the Romans. The Chinese write in vertical columns from above downwards. If they ever are compelled to write in a horizontal line, they write from right to the left. This is another indication that the abacus was not indigenous to China.

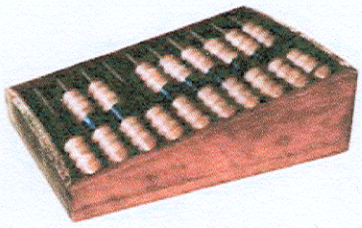


Figure 17.3

In the early Middle Ages there already existed in the Middle East kinds of wire-and-bead abacuses. The Armenian called this a *choreb*, the Turks a *coulba,* and the Russians, where it could still be seen in use till the end of the 20th century, a *stchoty*. For a 'modern' Russian *stchoty* see Figure 17.3

The suan-pan was a later development probably appearing in the twelfth century, and did not come into common use till the fourteenth century.
An old picture of a suan-pan is Figure 17.4 from the Suan Fa Thung Tsung of 1593 A.D.
The modern Chinese 'abacus' or suan-pan consist of an oblong wooden frame holding a number of vertically arranged rods, 9 or more, on which wooden beads slide up and down. A beam running across the frame divides the rods into two sections: the upper side or 'heaven' with two beads (altobeads) on each rod and the lower side or 'earth' with five (hypobeads) on each rod. See Figure 17.5
One altobead is equivalent to five hypobeads.
The value of a bead depends upon which column is assumed to be the unit's column.

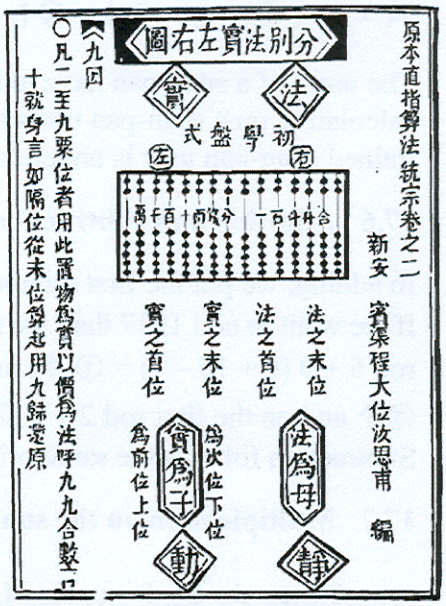


Figure 17.4

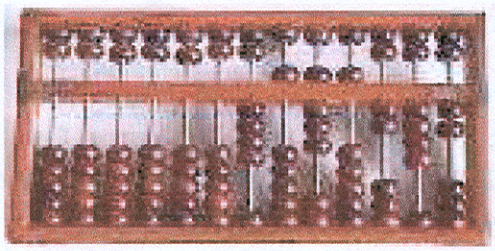


Figure 17.5

Thus, if we assume that the third column on the right-hand side of the frame to be the unit's place, one hypobead in the second column will be worth $1/10$; one in the first column will equal $1/100$ unit. The beam is the axis along which the active beads that form the computation are arranged, in Figure 17.5 that is 4585242. All the neutral or idle beads should be against the outher side.

17.5 Operating of and calculating rules for the suan-pan [2)]

By convention indicate the 5 hypobeads with ① to ⑤ and the two altobeads with ❺
Move with the thumb the ① to ⑤ upwards ↑ (raise), with the index finger the same downwards ↓ (lower) and with the middle finger the two ❺ ↑ and ↓. After centuries of use the next rules are proved to be the most efficient.

17 Rules for addition:

if on a rod the number of beads in the lower section already 4 are active than:

4 + 1: ❺↓ and ④↓ (+ 1 = + 5 - 4) 4 + 2: ❺↓ and ③↓ (+ 2 = + 5 - 3)

4 + 3: ❺↓ and ②↓ (+ 3 = + 5 - 2) 4 + 4: ❺↓ and ①↓ (+ 4 = + 5 - 1)

if on a rod already one of the altobeads are active (❺) than:

❺ + 6 = ❺↑ + ①↑ and on the next rod ①↑ (to forward ten = to move one hypobead

❺ + 7 = ❺↑ + ②↑ and on the next rod ①↑ on the next column to the left)

❺ + 8 = ❺↑ + ③↑ and on the next rod ①↑

❺ + 9 = ❺↑ + ④↑ and on the next rod ①↑

if on a rod already 9 beads are active than:

9 (= ④ + ❺) + 1 = ④↓ , ❺↑ + and on the next rod ①↑

9 (= ④ + ❺) + 2 = ③↓ , ❺↑ + and on the next rod ①↑ etc.

The users of a suan-pan have to know these rules by heart and apply these blindly. To learn calculating on a suan-pan takes more time compared to an electronic calculator…, but a trained suan-pan user is unbelievable fast and exact. See for this also §18.5.

17.6 Addition and subtraction on the suan-pan

In adding, we put the first number on the frame, e.g. 5612

If we want to add 1937 than we start at the fourth rod: one hypobead on ① (↑); on the third rod 6 + 9 (6 + 10 – 1) = ① ↑; on rod 4 (forward ten) and ① ↓ on rod 3; on the second rod ③↑ and on the first rod 2 + 7 (2 + 5 + 2) = ❺ ↓ en ② ↑. The sum is 7549

Subtraction follows the same principle with own rules.

17.7 Multipication on the suan-pan

Example: multiply 45 by 78.

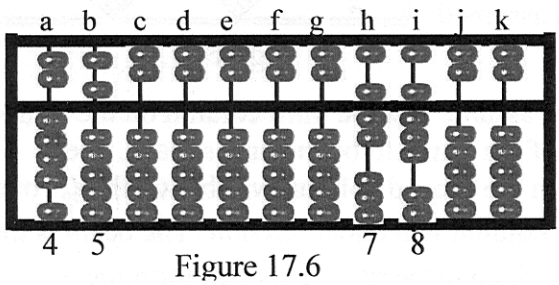


Figure 17.6

5 x 8 = ④(0)

4(0) x ~~8~~ = 32 (0)

③ ②

③ ❺+①

5 x 7(0) = 35 (0)

③ ⑤

❺+② ①

4(0) x ~~7~~ = 28(00)

② ⑧

③ ❺ ①

The multiplier (45) is set on the left-hand side and the multiplicand (78) on the right-hand side of the frame, leaving as many rods vacant on the right as the number of figures of the multiplier, see Figure 17.6

First: 5 (of 45) x 8 = 40; raise ④↑ on the second rod *j*. Next: 4 (40 of 45) x 8 = 32(0); cancel 8 on rod *i*; set 32 on rod 3 and 2: thus ③↑ on rod *i* and 4 + 2 = (+2 = +5 - 3) = ❺↓ and ③↓ on rod *j*. Next: 5 (of 45) x 7(0) = 35(0) on rod 3 and 2; thus on rod *i* 3+3 = (+3 = + 5 - 2) = ❺↓ and ②↓ and on *j* 6 + 5 = ❺↑ and on the next (*j*) ①↑ (forward ten). Next: 4(0) x 7(0) = 28(00); cancel 7 on rod 4; set 28 on rod 4 and 3: on rod 4 ②↑ and on rod 3: 7 + 8; (+8 = + 10 – 2): on rod 4 ①↑ (forward ten) and on rod 3 ②↓. The product is 3510

17.8 Division on the suan-pan (first if the divisor is less than 10) Example: 1497 by 2

The divisor (2) is set on the right-hand side of the frame, the dividend (1497) on the left-hand side, leaving one rod vacant (a) for the quotient, see Figure 17.7

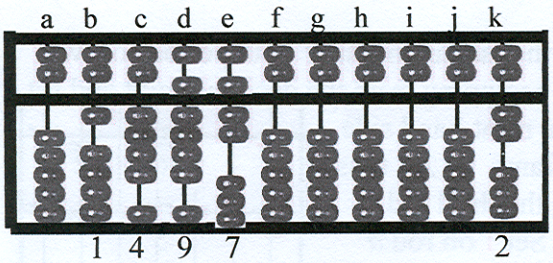


Figure 17.7

First ① (=1000) on b by 2 = 5(00); cancel ① on b and change it to ❺ on the same rod (one could ① on b replace by 2x ❺ on c; divided by 2 = 1 x ❺; set it one rod to the left on b). Next ④ by 2 = 2; cancel ④ on c and replace by ② on b; (on b now are ❺ + ②= 7, the first figure of the quotient.

Next 9 by 2 = $\frac{8+1}{2}$ first cancel ❺ + ③ on d and add ④ on c;
next cancel ① on d and replace by ❺ on d.

Next 7 by 2 = $\frac{6+1}{2}$ first cancel ❺ + ① on e and add ③ on d;
next cancel ① on d and replace by ❺ on e: the quotient is: 7 4 8 . 5

❺ ❺❺
② ④ ③

17.9 Division on the suan-pan (now if the divisor is more than 10) Example 1536 by 12

The divisor (12) is set again on the right-hand side of the frame, the dividend (1536) on the left-hand side, leaving one rod vacant (a) for the quotient, see Figure 17.8

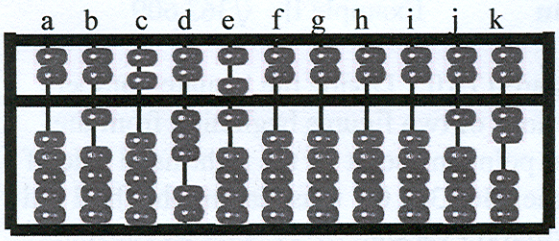
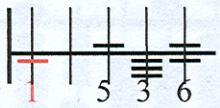
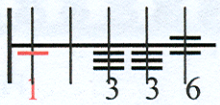


Figure 17.8

First: 1 of the dividend by 1 of the divisor:
1 by 1 = 1. Multiply this part of the quotient by the second figure of the divisor 1 x 2 = **2**;
set 1 on a, cancel 1 on b,
subtract the product **2** from 5 of the dividend 5 -**2** = 3;
from the dividend remains 336, to divide by 12.





The new left-hand figure of the dividend is 3 (336)
3 by 1(first figure of the divisor) = 3 and 3 x 2 (second figure) = **6**
because **6** is to large to be subtracted from 3 (from 336),
therefore 2 instead 3: set 2 on b (second figure of the quotient)
2 x 1 = **2**(00) and 2 x 2 (second figure) = 4(0)
336 - **2**(00) = 136 and 136 – 4(0) = 96 (96 is left undivided)
Next: 8 by 1 = 8 (9 is to large) 8 is third figure of the quotient
set 8 on c 9(0) – 8(0) = 1(0) 16 is left,
8 x 2 = 16 16 -16 = 0 the quotient is 128

17.10 Extracting the square root on the suan-pan

Example I: $\sqrt[2]{625}$

The square root of a number is found according to the Chinese method[3)], that means by help of repeated subtraction, as shown in Figure 17.9

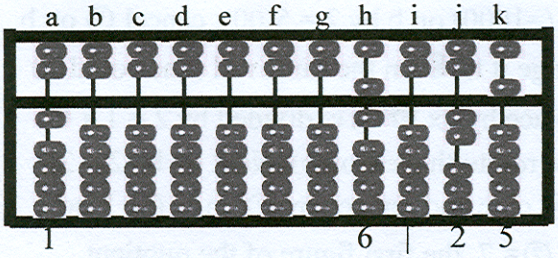


Figure 17.9

Divide the 'square number' into periods of two figures beginning from the decimal point and set it on the right-hand side of the frame 6 | 2 5 thus leaving the third rod *i* vacant. Set 1 on rod *a* (part of the root). Subtract 1 from 6 = 5(525) and add **2** to 1 = 3; subtract 3 from 5 = 2 (225). Now the 'root' 3 > first figure of the (remainder of the) square; multiply 3 by 10 = 30 and add **11** to 30 = 41.

Subtract 41 from 225 = 184.

Repeat the above steps with + **2** until the root ≤ the square:

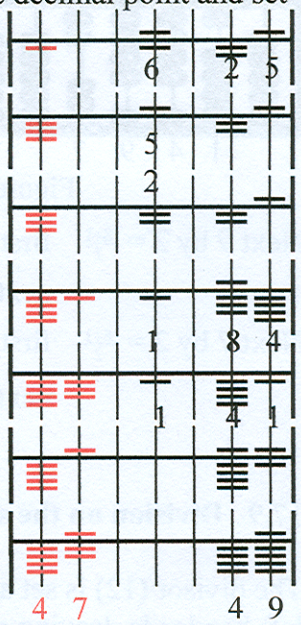
41 + **2** = 43 184 – 43 = 141

43 + **2** = 45 141 – 45 = 96

45 + **2** = 47 96 – 47 = 49

47 + **2** = 49 49 – 49 = 0

49 + ***1*** = 50 50 by ***2*** = 25, the root is 25



17.11 Extracting the square root on the suan-pan

Example II: $\sqrt[2]{363{,}609}$

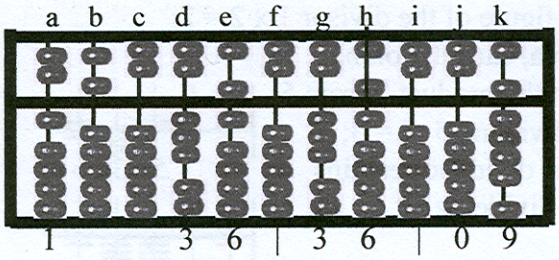


Figure 17.10

See Figure 17.10 Divide the 'square number' into periods of two figures beginning from the decimal point and set it on the right-hand side of the frame: 36 | 36 | 09 thus leaving the third rod *i* and sixth rod *f* vacant.

Set 1 on rod *a* (part of the root) and follow the same procedure as at example one:

36 – 1 = 35 1 + **2** = 3

35 – 3 = 32 3 + **2** = 5

32 – 5 = 27 5 + **2** = 7

27 – 7 = 20 7 + **2** = 9

20 – 9 = 11 9 + **2** = 11

11 – 11 = 0

Multiply 11 by 10 = 110 and add **11** to 110 = 121. Now 121 > 36, the second part of the square. Now we have to implicate the last part of the square in the calculation too: 3609. Now we have not to subtract 121 but 11 times 100 + **101** = 1201.

3609 – 1201 = 2408 1201 + **2** = 1203

2408 – 1203 = 1205 1205 + **2** = 1205

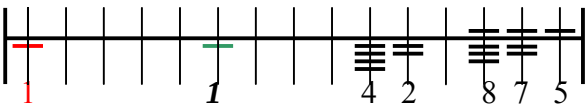
1205 – 1205 = 0

1205 + ***1*** = 1206 1206 by ***2*** = 603, the root is 603

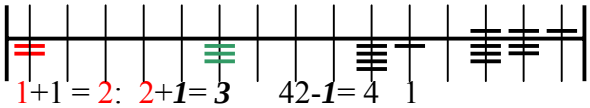
1 4 2 | 8 7 5

Figure 17.11

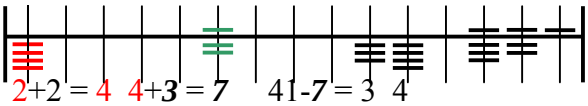
Root number square number cube number



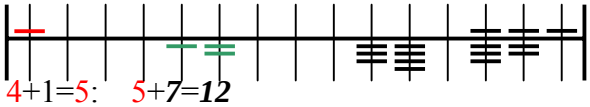
1 ***1*** 4 2 8 7 5



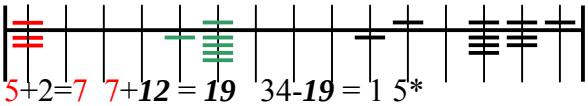
1+1 = 2: 2+***1***= ***3*** 42-***1***= 4 1



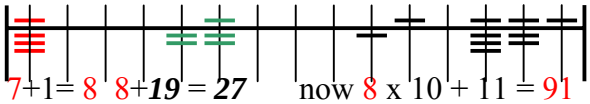
2+2 = 4 4+***3*** = ***7*** 41-***7*** = 3 4



4+1=5: 5+***7***=***12***



5+2=7 7+***12*** = ***19*** 34-***19*** = 1 5*



7+1= 8 8+***19*** = ***27*** now 8 x 10 + 11 = 91

42 – ***1*** = 41

1 + 1 = 2 2 + ***1*** = ***3***

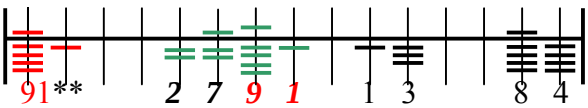
2 + 2 = 4 4 + ***3*** = ***7*** 41 – ***7*** = 34

4 + 1 = 5 5 + ***7*** = ***12***

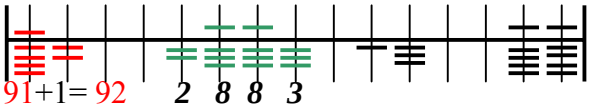
5 + 2 = 7 7 + ***12*** = ***19*** 34 – ***19*** = 15

*now 15 is less than ***19***; from here we have to calculate with 15,875 (incl. the 2nd period)

7 +1 = 8 8 + ***19*** = ***27*** 8 x 10 = 80 80 + 11 = 91

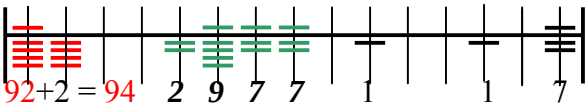


91** ***2 7 9 1*** 1 3 8 4

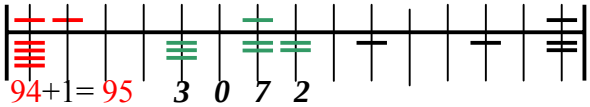


91+1= 92 ***2 8 8 3***

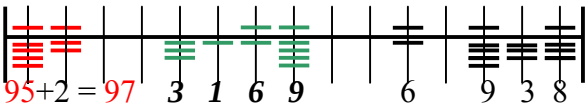
** put 91 behind 27: 15,875 – ***2791*** = 13,084 92 + ***2791*** = ***2883***



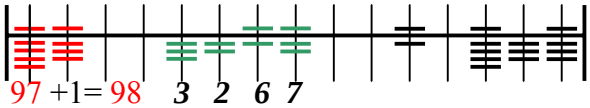
92+2 = 94 ***2 9 7 7*** 1 1 7



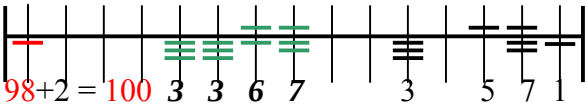
94+1= 95 ***3 0 7 2***



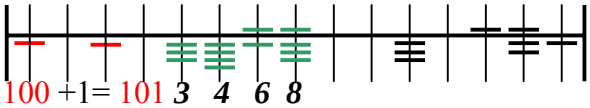
95+2 = 97 ***3 1 6 9*** 6 9 3 8



97 +1= 98 ***3 2 6 7***



98+2 = 100 ***3 3 6 7*** 3 5 7 1



100 +1= 101 ***3 4 6 8***

91 + 1 = 92 92 + ***2791*** = ***2883***

92 + 2 = 94 94 + ***2883*** = ***2977*** 13,084 – ***2977*** = 10,107

94 + 1 = 95 95 + ***2977*** = ***3072***

95 + 2 = 97 97 + ***3072*** = ***3169*** 10,107 – ***3169*** = 6938

97 + 1 = 98 98 + ***3169*** = ***3267***

98 + 2 = 100 100 + ***3267*** = ***3367*** 6938 – ***3367*** = 3571

100 + 1 = 101 101 + ***3367*** = ***3468***

101 + 2 = 103 103 + ***3468*** = ***3571*** 3571 – ***3571*** = 0

103 + 2 = 105 105 by 3 = 35 the cube-root is 35

17.13 Extracting the square root on the suan-pan

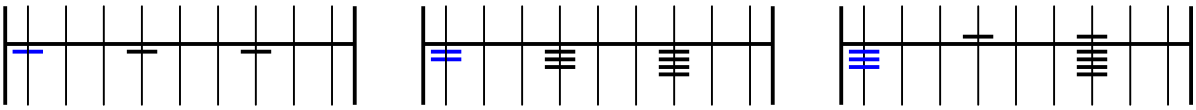
Example I: $\sqrt[2]{625}$

Following a method developed by the author, a square root can be found with a suan-pan, using repeated addition instead of subtraction, analogical to extracting roots on the CURTA, see §14.2.1.5

Dividing 6 | 25 into periods of two figures indicates a root of two figures:
set **1** for the root on *a*, ***1*** for the added number on *d* and 1 for the square on the 3rd rod from the right.
Root (equal to the number of steps, also to the number of turns of the handle of the CURTA)

added number **root** square

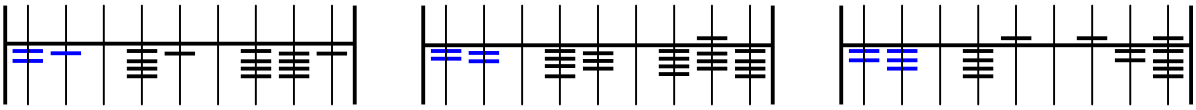
a b c d e



1(0) ***1*** 1(00) **2**(0) ***3*** 4(00) **3**(0) ***5*** 9(00)

Step 1 step 2 at the next step ***3*** + 2 should be ***5***
1 + 2 = ***3*** and 4(00) + ***5*** = 9(00); for
3 + 1(00) = 4(00) 9(00) > 6(25) instead of that:

3 + 1 = ***4*** ***4*** x 10 = ***40*** (x 10 = carriage of the CURTA on position 2) and ***40*** + 1= ***41***

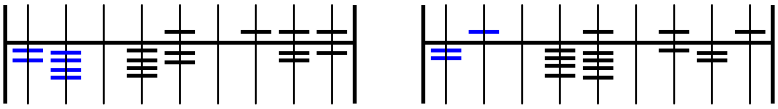


2 1 ***4 1*** 4 4 1 **2 2** ***4 3*** 4 8 4 **2 3** ***4 5*** 5 2 9

step **3** step **4** step **5**

4(00) + ***41*** = 441 441 + ***43*** = 484 484 + ***45*** = 529

41 + 2 = ***43*** ***43*** + 2 = ***45***



2 4 ***4 7*** 5 7 6 **2 5** ***4 9*** 6 2 5

step **6** step **7**

529 + ***47*** = 576 576 + ***49*** = 625

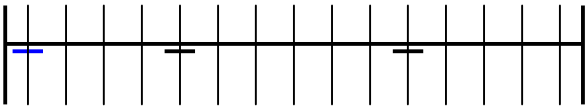
45 + 2 = ***47*** ***47*** + 2 = ***49***

to the right now we have 625, the square to be extracted, thus the root is **25**

17.14 Extracting the square root on the suan-pan Example II: $\sqrt[3]{363{,}609}$

Dividing 36 | 36 | 09 into periods of two figures indicates a root of three figures (on *a*, *b* and *c*): set **1** for the root on *a*; ***1*** for the added number on *e* (let d vacant) and 1 for the square on the 5th rod from the right: (we will need a suan-pan of at least 15 rods)

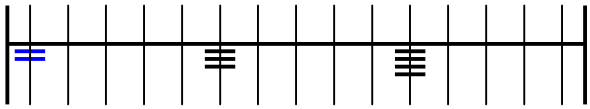
a b c d e f g



1(00) ***1*** 1(0,000)

step **1**

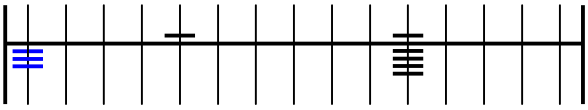
1 + 2= ***3***



2(00) ***3*** 4(0,000)

step **2**

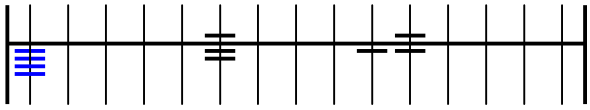
3 + 1(0,000) = 4(0,000)



3(00) ***5*** 9(0,000)

step **3**

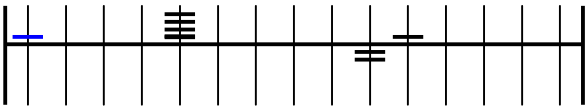
3 + 2= ***5*** ***5*** + 4(0,000) = 9(0,000)



4(00) ***7*** 16(0,000)

step **4**

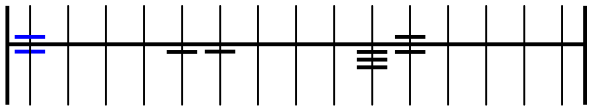
5 + 2 = ***7*** ***7*** + 9 = 16(0,000)



5(00) ***9*** *25*(0.000)

step **5**

7 + 2= ***9*** ***9*** + 16(0.000) = 25(0.000)



6(00) *11* 36(0.000)

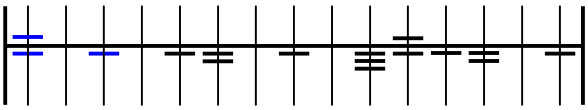
step **6**

9 + *2*= ***11*** ***11*** + 25 = 36(0.000)

36 is equal to the first two figures of the square:

11 + 1 = ***12*** ***12*** x 10 = ***120*** ***120*** + 1= ***121***,

121 is larger than 36: that means the second figure of the root = 0, therefore ***12*** x 100 = ***1200***



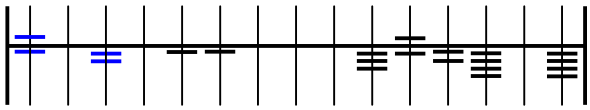
601 ***1201*** 361,201

step **7**

1200 + 1 = ***1201***

1201 + 2 = ***1203***

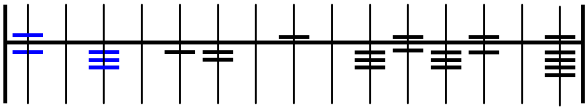
1201 + 360,000 = 361,201



602 ***1203*** 362,404

step **8**

1203 + 361,201 = 362,404



603 ***1205*** 363,609

step **9**

1203 + 2 = ***1205*** ***1205*** + *1* = ***1206*** ***1206*** : *2* = **603**

1205 + 362,404 = 363,609 = the square thus **603** is the root

17.15 The logarithms in China

During the 17th century, the Jesuit order was sending its technically trained members around the world as missionaries both for the Christian faith and the wonders of European technology. From Rome, Gaspard Schott (see section 12.1) gathered and exchanged all information about the newest technical developments with all Jesuits all over the world.

In this way the logarithmic table of Ezechiël De Decker (and Napier's rods, see section 18.1) arrived in China.
Figure 17.12 is the top right corner of a page of a Chinese logarithmic table:

Lg 87,501 (=) 4 (.) 9420130164

8 7 5 0 1 ; 4 9 4 2 0 1 3 0 1 6 4

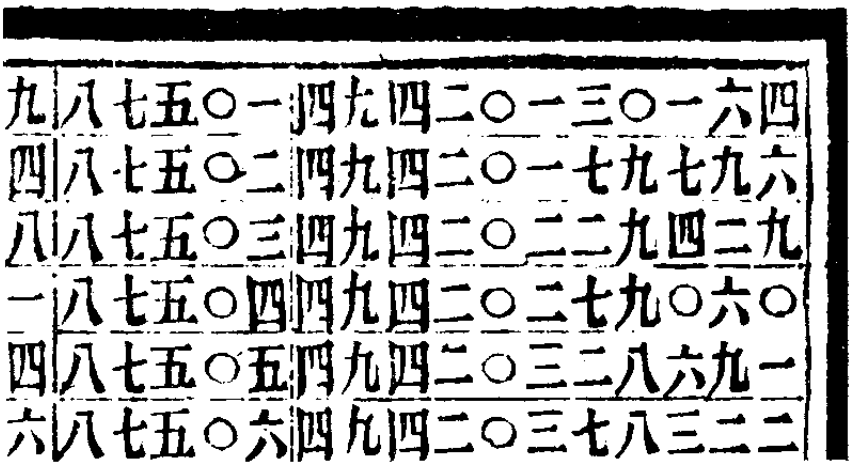


Figure 17.12

One of the three Chinese symbols for numbers is used. One can recognise the 1, 2, 3 and the cipher in the uppermost line.

The use of the suan-pan in China became known in Korea step by step and from there it spread to Japan.

17.16 Reckoning in the binary system on a suan-pan [5]

The suan-pan can work – in addition and subtraction – as well in decimal as it can in the binary system. What make the suan-pan useful for binary calculations are the two beads in each column above the crossbar. This 'heaven' section is nicely suited to this kind of reckoning.

In binary there are only two digits: 0 and 1.
The larger of these two digits is 1.
If we add another unit to this 1, it splits into two digits to become 10.
A digit alone, in binary, is still unit. But a digit to the left of it represents *twos*; a third digit to the left of that represents two times two, or *fours*; a fourth digit to the left of that represents two times two times two, or *eight's*, and so on, see Figure 17.13

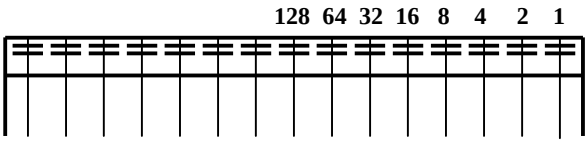


Figure 17.13

Before calculating begins, we clear the suan-pan by sending all heaven beads up to the top of the frame, see Figure 17.13
Let's add the two numbers of § 11.12.2:
155 = 128 + 16 + 8 + 2 + 1 or 10011011 and
74 = 64 + 8 + 2 or 1001010.
For 155 we bring one heaven bead down to the crossbar in the 1st, 2nd, 4th, 5th and 8th column, see figure 17.14

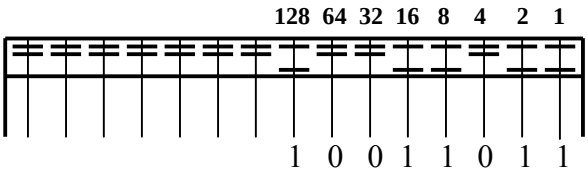


Figure 17.14

Now we feed in each column the second number (74): bring one heaven bead down to the crossbar in the 2nd, 4th and 7th column. With both numbers fed in, we should have the result of Figure 17.5

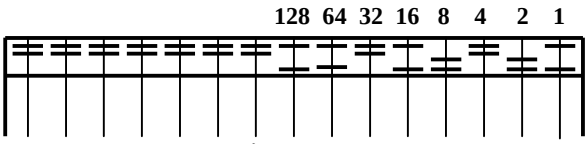


Figure 17.15

The second and fourth columns now have two beads at the crossbar. We exchange these two beads for one in the next column: bring these two beads back to the top and send one of the heaven beads in the next column down to the crossbar. Now we should have the result of figure 17.16

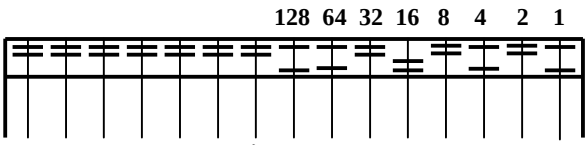


Figure 17.16

Now we repeat this for the fifth column. Result: 11100101 or 128 + 64 + 32 + 4 + 1 = 229
See Figure 17.17

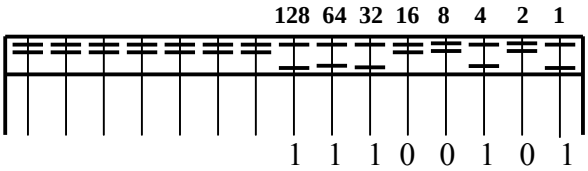


Figure 17.17

1) Takashi Kojima
The Japanese Abacus Its Use and Theory

2) Lau Chung Him, B.A. Hong Kong 1958
The principles and practice of the Chinese abacus

3) The Fundamental Operations in Bead Arithmetic.
How to use the Chinese abacus

4) Carl B. Boyer / Uta C. Merzbach
A History of Mathematics

5) Jesse Dilson
The abacus The World's First Computing System

Chapter eighteen, *Japan*

18.1 Sangi, J apanese counting-rods and rod-numerals

Shortly after the Koreans had introduced Buddhism to Japan in 552 A.D., Korean scholars also brought the Chinese calendar, astrology and the knowledge about the chou, the Chinese counting rods and their usage into Japan. The Japanese called those numbers sangi or san-ju. The Japanese changed the shape of the cylindrically rods by adding a square part in the centre in order to prevent them for rotating. These sangi rods were made of cherry wood and were about 6 mm in diameter and ca. 40 mm long. Figure 18.1 is a picture of using these rods.

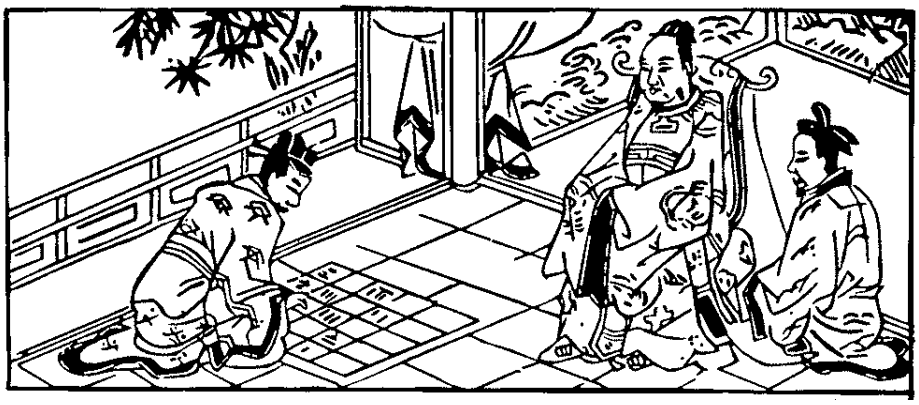


Figure 18.1

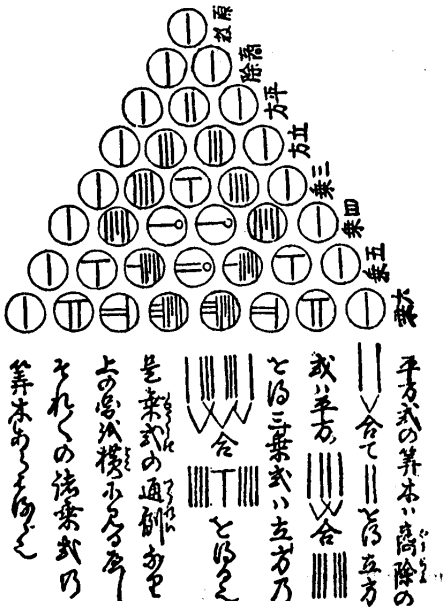


Figure 18.2

An example of the use of the sangi numerals is in a socalled triangle of Pascal from 1781, see Figure 18.2

Clearly is to see: |||+|||=⊤and |||| + ⊤ = —o

Also —o +—o = ═o

The small circles to these 10 and 20 are Japanese additions to the written use of the sangi numerals.

On the Japanese rods of Napier from the begin of the 19th century, a different set of numerals is used, see Figure 18.3.

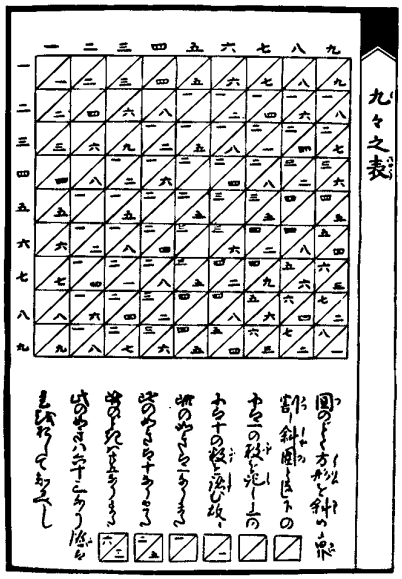


Figure 18.3

In China and Japan these rods already appeared some years after the publication by Napier (also see section 17.15 on the logarithm table in China).

18.2 The Soroban

The Chinese concept of a wire-and-bead abacus first appeared in Korea around 1400 A.D. Although the soroban, as this bead-abacus was called in Japan, did not come into popular use in Japan until the 17th century, there is no doubt that it must have been known to Japanese merchants at least a couple of centuries earlier. Like the present-day Chinese suan-pan, the soroban had two alto beads and 5 hypo beads. But toward the close of the 19th century it was simplified by reducing the alto beads to one, and finally around 1920 it acquired its present shape by omitting yet another bead, reducing the hypo beads from five to four.

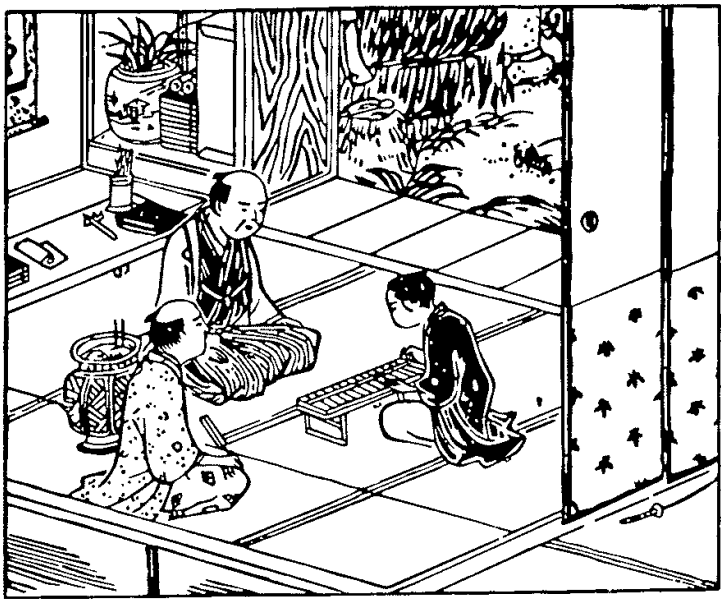


Figure 18.4

A picture of a student who takes a course on the soroban is shown in Figure 18.4.

The most common type of soroban has 21 keta's (Japanese for wires), other types have 13 and 17 to 27 and 31 keta's.
The distance between two keta's is 12 mm and the distance a bead at most can be moved is 5 mm. The distance of beads next to an other only is 1 mm.

Not a device for a clumsy individual! See Figure 18.5.

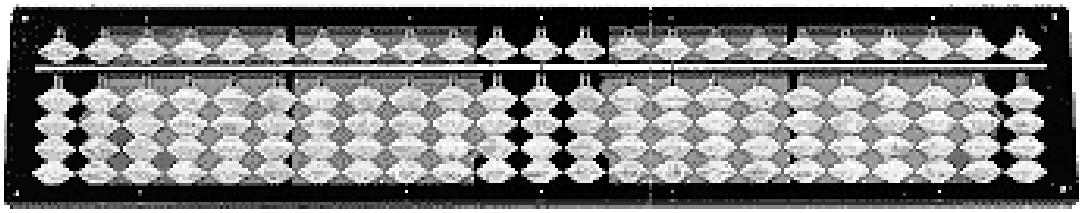


Figure 18.5 a soroban with 23 keta's

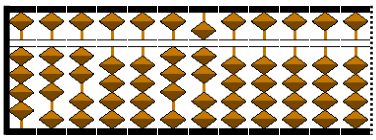
For convenience in calculation the beam is marked with a unit point at every third keta. For example, select any keta near the centre of the soroban, which is marked with a unit point, and call this the unit keta of the problem. In clearing the soroban for use, hold the left end with your left middle finger on its upper edge and your left thumb on its lower edge, and move all beads down by slanting the upper edge toward your body. After levelling the soroban again, raise all 5-unit beads by moving the right index finger from left to right along the upper edge of the beam. When calculating on the soroban, use two fingers: the right index finger to move the alto beads up and down and to move hypo beads down, while using the thumb only to move hypo beads up.

18.3 Multiplication on a soroban

Set the multiplier on the left of the soroban and leave two keta's vacant between multiplier and multiplicand; the product is set next to the multiplicand.

Example 37 x 432 first 4 x 7, next 3 x 7 and 2 x 7

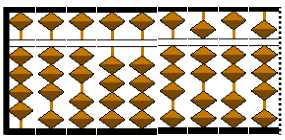
a b c d e f g h i j k l



4 3 2 *3 7*

step 1 set 432 and 37

a b c d e f g h i

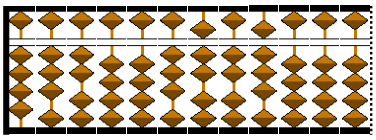


4 x *7* 2 8

step 2 4 x *7* = 2 8

at step 2 one don't have to memorise that 4 is 400

a b c d e f g h i j k l

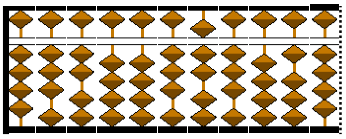


step 3 2 8

3 x *7* = 2 1

at step 3 add 2 to 8 by clearing 8 on keta i and forward ten by moving up 1 on keta h.

a b c d e f g h i j k



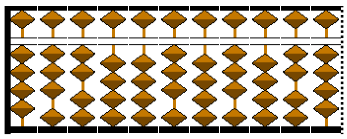
step 4 3 1

2 x *7* = 1 4

set 4 on k and add 1on j

after step 4 cancel *7* on g

a b c d e f g h i j k



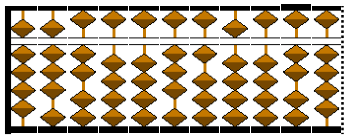
step 5 3 2 4

4 x *3* = 1 2

set 1 on g

add 2 to 3 on h

a b c d e f g h i j k



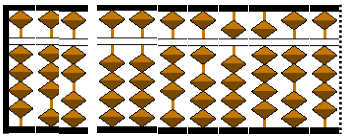
step 6 1 5 2 4

3 x *3* = 9

set 9 on i

a	b	c	d	e	f	g	h	i	j	k
4	3	2	0	0	3	7	0	0	0	0
step 2							2	8		
step 3								2	1	
step 4									1	4
4	3	2	0	0	3	0	3	0	2	4
step 5						1	2			
step 6								9		
step 7									6	
4	3	2	0	0	0	1	5	9	8	4

a b c d e f g h i j k

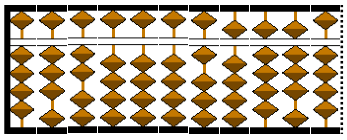


step 7 1 5 9 2 4

2 x *3* = 6

after step 7 cancel *3* on f

a b c d e f g h i j k



step 8 1 5 9 8 4 x

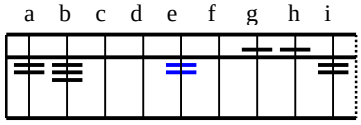
add 6 to 2 on j

18.4 Division on the soroban

Set the divisor ( ) on the left of the soroban and leave four keta's vacant between divisor and dividend;
the quotient is set left to the dividend, leave 1 keta vacant:

Example: 552 ÷ schematically represented:

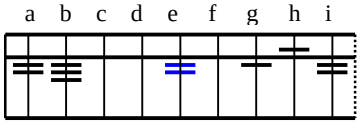
a b c d e f g h i



divisor 2 dividend

step 1: **5**(**52**) ÷ ( ) = 2

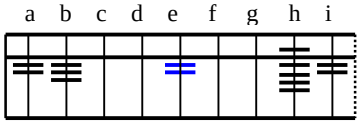
a b c d e f g h i



2 **1 5 2**

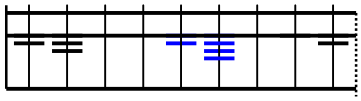
step 2: x 2 = *4* and **5**(g) – *4* = **1**

a b c d e f g h i



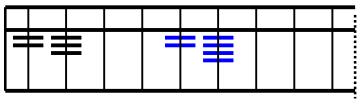
step 3: x 2 = *6* **9 2**

and **15**(gh) – *6* = **9**(h)



step 4 **9**(2) ÷ ( ) = 4

x 4 = *8* and **9** – *8* = **1**(h)



step 5: x 4 = *12*

and **12** – *12* = 0 quotient = 24

To learn the principles of calculating on both the suan-pan and the soroban is as easy as to learn the rules of the game of chess. To become a trained calculator needs years of practice. In the hands of a trained operator, the soroban is a powerful and sophisticated aid to computation.

The Abacus Committee of the Japan Chamber of Commerce and Industry has made minute investigations concerning the potentialities of the Japanese soroban. The Committee has acted as judge of the semi-annual examination for abacus operator's licenses since these examinations were initiated in 1931. Contests were organised and grades could be obtained. After six months practice of one hour a day the third-grade, after an another half year the second-grade and after an other full year the first-grade could be obtained.

18.5 Contest between the J apanese soroban and an electric calculating machine[1)]

During the American occupation of Japan from 1945 the Yankees looked sceptical to the actions of the Japanese on their archaic calculating instrument.

An exciting contest between the Japanese soroban and the electric calculating machine was held in the Takarazuka Theatre in Tokyo on November 12, 1946, under the sponsorship of the U.S. Army newspaper, the *Stars and Stripes*, between:

- the Japanese representative Mr. Kiyoshi Matsuzaki, (22 years, 7 years experience) and a champion operator of the soroban in the Savings Bureau of the Ministry of Postal Administration on a soroban of 25 cent and
- the American representative of the calculating machine Private Thomas Nathan Wood, (22 years, 4 year experience) of the 240th Finance Disbursing Section of General MacArthur's headquarters, who had been selected in an arithmetic contest as the most expert operator of the electric calculator in Japan on a calculator of $ 700!

The contest consisted of 5 series tasks:

Addition: 50 numbers each consisting of from 3 to 6 digits;
Subtraction: 5 problems, with minuends and subtrahends of from 6 to 8 digits;
Multiplication: 5 problems, each containing 5 to 12 digits in multiplier and multiplicand;
Division: 5 problems, each containing 5 to 12 digits in divisor and dividend;
Composite problem: 30 additions, 3 subtractions, 3multiplications and 3 divisions.

The Japanese scored a total of 4 points as again 1 point for the electric calculator!

In reporting the contest, the *Stars and Stripes* remarked: "The machine age took a step backward yesterday at the Ernie Pyle Theatre as the abacus, centuries old, dealt defeat to the most up-to-date electric machine now being used by the United States Government…The abacus victory was decisive."

If numbers do not contain over six digits, the soroban manipulator can halve the time of the operation by relying in part upon *mental calculation*. This method consists in mentally visualising a soroban and working the problem out by standard methods on the imaginary instrument. It is rather like those master chess players who can defeat their opponents without even seeing the board.

1) Takashi Kojima
The Japanese Abacus Its Use and Theory Tokyo, 1954

Chapter nineteen, *The Mayas*

The Maya civilisation, reached the height of its development during its Great or Classical Period, from the 4th to the 10th century A.D. It covered Honduras, central and southern Mexico, Guatemala, and the western portions of Honduras and El Salvador.
Later they left their cities and moved to the Yucatan Peninsula. There under the 'New Empire', religion, science and art flourished once more until the 14th century.
In the 16th century the Spaniards, commanded by Hernández Cortéz, eagered for gold and by instigation for conversion, nearly destroyed the whole Maya culture.
In his *Relacion de las cosas de Yucatan*, Bishop Diego de Landa in 1541 tells of the burning of many Maya books, written on long strips of bark or parchment, folded like a screen. Three of these books have survived and are known as the 'codices':
- The Troano-Cortesian Codex, 42 en 70 pages, together 6,82 meter at Madrid,
- The Peresianus Codex, 1,45 meter at Paris and
- Dresden Codex, 3,56 meter, now in the Sachsische Landesbibliothek at Dresden.

19.1 One page of the Dresdener Codex

This codex is copied in the 9th century, the original is 3 to 4 centuries older, see Figure 19.1.

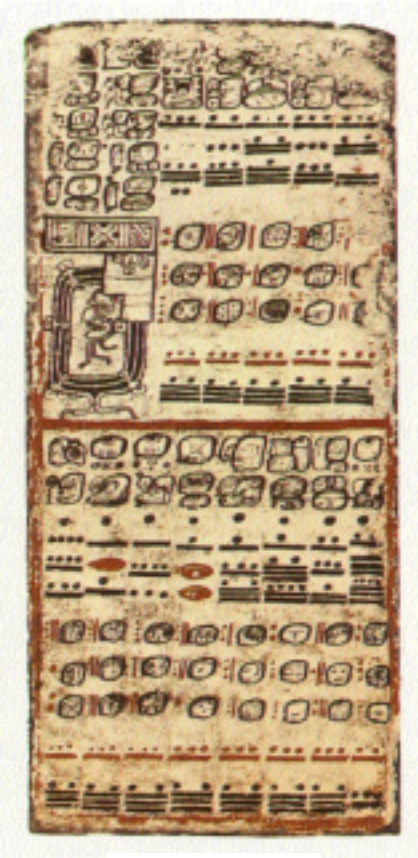


Figure 19.1 [2)]

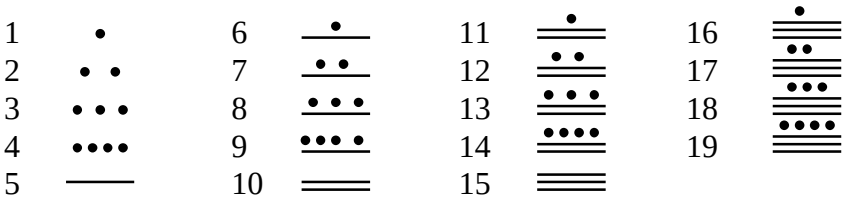
Later, Diego de Landa, as Bishop of Mérida, as 'penance' and for fear of reprisals paid attention to the rest of his live by recording, written by Maya in the Maya language, but in Spanish script, as much as possible of the Maya culture. Diego de Landa especially was interested in astronomy, calendar and numbers: by this it looks as if the Mayas had an obsession for these subjects.

Some time during the 4th or 3rd centuries B.C., the Maya priests for the first time in the history devised a system of numeration by position, involving the conception and use of the mathematical quantity of zero. The Mayan system of notation required only three symbols: a dot • (pebble?) was used for one and a bar —— (stick?) for five. Larger numbers were represented by a combination of these symbols up to 19, see Figure 19.1. In their system the values of the positions increase by twenties from bottom to top. By placing a shell = zero in the first or lowest position to denote 0 units of the first order and a dot in the second position to denote 1 unit of the second order, the number 20 was written. In counting everything the Maya followed the vigesimal progression consistently except in counting time, where the Maya introduced a single variation, in order to make the period of their third order agree in length with their calendar year as nearly as possible.

The third order of the Maya vigesimal calendar system, the TUN (see section 19.3), therefore, was composed of 18 (instead of 20) UNIALs, or 360 (instead of 400) KINs

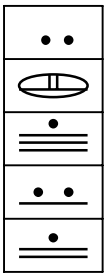
19.2 The vigesimale system of the Maya

The numbers of 1 to 19 can be written as:



For calculations for the calendar or time only the third position contained 18 x 20 = 360 units in stead of 20 x 20 = 400 so the value of each position respectively was as follows:

1 20 360 7.200 144.000 etc. or only for time and calendar:



$2 \times 360 \times 20^2 =$	288.000
$0 \times 360 \times 20 =$	0
$16 \times 360 =$	5760
$7 \times 20 =$	140
$11 \times 1 =$	11
	293911

1 20 400 8000 16000 etc. Example: for counting everything else:

$2 \times 20^4 =$	320.000
$0 \times 20^3 =$	0
$16 \times 20^2 =$	6400
$7 \times 20 =$	140
$11 \times 1 =$	11
	326551

19.3 The 'Haab' or civil or secular 365-day calendar [1)]

Time as well as dates were expressed in:

- KINs = days
- UINALs = months of 20 days
- TUNs = years of 18 months = 20 x 18 = 360 days, filled up with 5 days
- KATUNs = cycles of 20 years
- BAKTUNs = cycles of 20 katuns = 400 years
- PICTUNs = cycles of 20 baktuns = 8000 years etc.

19.4 Gods as hieroglyphs

Each of these units was represented by a weight bearing on the back of one of the deity head glyphs of the time. These gods, so the date belonged to an event to remember, were expressed as hieroglyphs, see Figure 19.2

KIN UINAL TUN KATUN BAKTUN

Figure 19.2

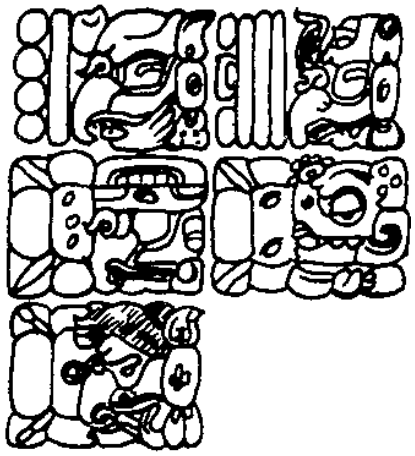


Figure 19.3
The upper section of the stela A at Quiriguà in Guatemala

These dates in hieroglyphs on monuments were represented by a number as prefix in a specific order, see Figure 19.3 These numbers were composed of 1 to 4 points with a value of 1 per point and 1 to 3 vertical lines, each line with a value of 5, so maximal ••••
═══ =19 for the first 19 days of a month, more was not needed for adding 1 day and the result was 1 month = •

The god who had to bear the weight of zero days could be missed, but for two reasons this was not possible.

The combination of aesthetic care of the more or less fixed lay-out of the hieroglyphs on monuments and the mystical fear to leave out a deity made the invention of the zero as prefix to a hieroglyph necessary and this combination also lead unavoidable to the use of the positional principle.

These stelea were constructed by the Mayas every 20 years and span at least 15 centuries.

In Figure 19.3 one can read: 9 BAKTUNS = 9 x 144.000 = 1.296.000 days
17 KATUNS = 17 x 7.200 = 122.400 days and
zero TUN, UINAL and KIN, total 1.418.400 days

To determine a date the Maya assumed of a start-date corresponding on 12 August 3113 B.C. (on Gregorian calendar) which would mean that the stela was constructed on 24 January 771 A.D.

19.5 The 'Tzolkin' or sacred year calendar

Besides the Haab, there was the 'Tzolkin' or sacred year of 260 days. The 260 days of the sacred year were formed by prefixing the numbers 1 to 13 inclusive to the twenty Maya gods or day-glyphs, (20 months of 13 days = 260 days), the names of which are given below:

1) Ik	6) Manik	11) Eb	16) Cahan
2) Akbal	7) Lamat	12) Ben	17) Eznab
3) Kan	8) Muluc	13) Ix	18) Cauac
4) Chicchan	9) Oc	14) Men	19) Ahau
5) Cimi	10) Chuen	15) Cib	20) Imix

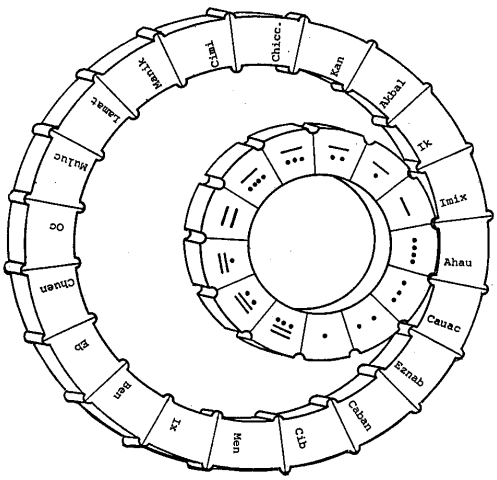


Figure 19.4

If the first day of the year is 1 Ik, then the second day is 2 Akbal etc., the 13th day is 13 Ix, the 14th day is 1 Men, the 15th day is 2 Cib, the 20th day is7 Imix, then the 21st day again is 8 Ik etc.

In order to show clearly how the 13 numbers were combined with the twenty patron saints (guardian deity) let us represent them graphically as two Tzolkin cogwheels[2)] see Figure 19.4 the day named 4 Ahau

The 261st day having the same designation Ik as the first day, that is one complete round of 260 different named days – the Tzolkin or 'count of days'.

Astronomy was totally dedicated to religion and astrology. The religion knew uncountable gods: the sun, the moon, the planets, all stars, nature gods as of the maize, bee's etc. A Maya took his given name from the day of the Tzolkin upon which he / she happened to have been born. For every Maya the day of birth determined the destiny. It connected the newborn with the God of the day of the Tzolkin or sacred year calendar.

19.6 The combination of the calendars, the Tun for 'long counts'

To determine a day in the Maya calendar both Tzolkin (Fig. 19.4) and Haab, the 365 days calendar had to be combined, see Fig. 19.5. The Haab of which the names of the gods are:

1) Pop	5) Tzec	9) Chen	13) Mac	17) Kayab
2) Uo	6) Xul	10) Yax	14) Kankin	18) Cumhu
3) Zip	7) Yaxkin	11) Zac	15) Muan	19) Uayeb
4) Zotz	8) Mol	12) Ceh	16) Pax	

Figure 19.5 is the representation of date 4 Ahau 8 Cumhu. The Haab month started at day 0 (zero), so 0 Pop is the first day of the month; 1 Pop is to compare to: it is one o'clock, the first hour has passed.

The least common multiple of 260 and 365 is 5 x 52 x 73 = 18,980 days therefore, if the Tzolkin wheel will make 73 complete revolutions, the Haab wheel will make 52 (years) complete revolutions before the combination 4 Ahau 8 Cumhu will return.

Compared to their European contemporaries the Maya tower above them concerning the calendar, arithmetic and astronomy.

The Mayan cities only were religious and astronomical centres in which the nobles and the priests lived. These priests were excellent astronomers

Tzolkin wheels

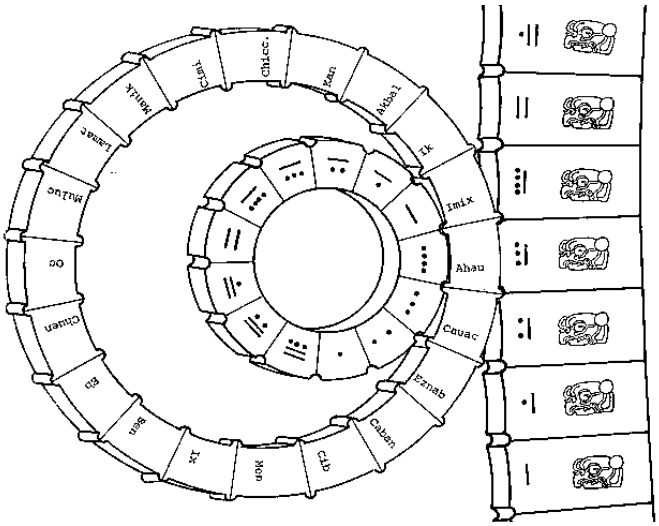


Figure 19.5 Haab wheel

The Mayan estimate of the duration of the solar year expressed in modern terms, is 365,242 days; the currently accepted value is 365,242198 days, that is more exact than our Gregorian calendar! A similar degree of accuracy was obtained for the average duration of a lunar 'month': according to the Mayan astronomers 149 lunar months lasted 4400 days. This is equivalent to an average lunar month of 29,5302 days – with our present-day knowledge and technology the figure we get is 29,53059 days! Of Venus they determined the orbit time at 584 days. The real orbit time is 583,92 days!

1) Henrietta O. Midonck
The Treasury of Mathematics / The Maya Civilisation

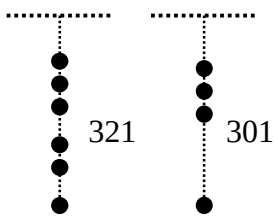
2) Erich von Daniken
De dag dat de goden kwamen

Chapter twenty, The *Incas*

At its height during the last decades of the 15th century A.D., the Inca Empire occupied an area that today would include all of Peru and parts of Bolivia, Chile, Ecuador and Argentina and there lived a culturally diverse population of about 6 million. Although it was a well-organised society, they had no system of writing, money was unknown and barter was the means of trading, also the wheel and beasts of burden were unknown.
Nevertheless the empire was excellently ruled from the capital city Cuzco. The required data and information for this were in the form of knots or *quipus*, recorded in cords.

20.1 Quipus

Quipu is a Quechua (the language of the Incas) word meaning, 'knot'. With a quipu a combination of several cords of wool or cotton is meant. From a thicker main cord pendent cords are attached like a fringe, equipped with a number of knots. The quipu was basically a device for recording numbers in a decimal base system. According to their position, the knots could indicate units, tens, hundreds, thousands and ten thousands, the highest-valued digit closest to the main cord. If necessary higher values could be used for in Quechua they could denominate any number.



← Main cord, with pendent strands, vary in number from one to even 2000

The absence of a knot indicated zero in any of the positions! Thus the Inca's had a decimal place value system in a highly developed form and used zero!

To construct a quipu, a thicker cord for the main cord of twice the length desired is to spin the cord, then double it with a loop at one end, as seen in Figure 20.1 (a) and twist the two strands together. The pedant strands with information were also doubled and twisted and attached to them with a knot (b). Short subsidiary cords are attached to the pendant strands, upon which are indicated numbers that disturb the main count of the quipu. To record particular information special knots were used, see Figure 20.1[2)] The colour of the cords was given special meanings: white represented silver, yellow gold and red soldiers or war. The information recorded in quipus was collected and delivered throughout the empire by runners or *Chasquis*.

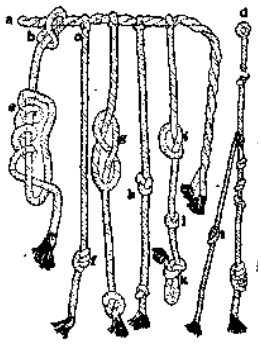


Figure 20.1

20.2 Chasquis or runners

Cieza de León, one of the soldiers of the Spanish conquerors, began keeping a chronicle in 1547 – fifteen years after the Spanish conquest – during 3 years. It provides a fascinating and reliable account, both of the flora and fauna. He also described the 'imperial highways' where about every 6,5 km small post-houses were located, where trained runners, called *chasquis*, were stationed in pairs, who carried messages across the mountainous terrain. Running at top speed and handling their quipus on from one runner to the next, they could transmit a message to the imperial capital Cuzco from 500 km away in 24 hours.
Frequently the report, which was coded in the quipu, was supplemented by carefully memorised oral messages. In a community without writing one is trained to memorise. This talent deteriorated with the spread of the knowledge of writing.

The quipu not only was used for mathematical accounting but also for memoranda of historical events, lists of kings, ceremonies, negotiations for peace or war etc.

20.3 Quipu-camayus or quipu keepers

Each district under the rule of the Incas had its own specially trained individuals *quipu-camayu*, whose duty it was to record by means of the quipu amounts of tribute, the size and type of flocks, items relating to the administration of justice, etc. At giving an account of the natives of each village, on the first string they put only men of 60 and upwards, in the second those of 50, in the 3rd those of 40 and so on. The women were counted in the same order. If the main strand of men of a certain age had reference to married people, the supplementary strand gave the number of widowers of the same age in that year. These accounts were made annually and only related to one year.

20.4 An Inca counting-board?

Guaman Poma de Ayala, a Peruvian, in about 1600 sent a 1179-page letter to the king of Spain. This is one of the most important sources for information about the Incas. It contains a series of illustrations in which seven of these drawings shows people carrying quipus, see Figure 20.2 (from the *Codex Péruvien* at the Institut d'Ethnologie at Paris) for one of this drawing: the Chief Treasurer to the Inca with a 'blank', un-knotted quipu.[1)]

Garcilasso de la Vega, (b. 1559) son of a Spanish cavalier and an Inca princess, (a niece of the last Inca king), describes in the *Royal Commentaries* the organisation of the Inca state and the Indian way of life.
He laid great stress on the gentle nature of the people, their honesty and fair dealing.

Before the result of a calculation could be recorded on a quipu, first this calculation had to be made.

José de Acosta, a Spanish priest who lived in Peru from 1571 to 1586 wrote in 1596: "To see them carry out a very difficult computation with maize kernels is a perfect joy. They move one kernel here and there and the fact is that they are able to complete their computations without making the smallest mistake". In calculation they are superior to us.
These calculations could be done on a board like the illustration at the bottom left of Figure 20.2.

Figure 20.2

1) George Gheverghese Joseph
The Crest of the Peacock Non- European Roots of Mathematics

2) Henrietta O. Midonick
The Peruvian Quipu in The Treasury of Mathematics

Index of Operations

Addition and Subtraction as described in the peceding chapters can be performed in many different ways:

on the Greek Salamis counting board, see § 3.5;
on the Roman hand-abacus, see § 4.2;
on a Hindu (dust)abacus, see § 6.8;
on the lines, see § 8.17.2 and 8.17.3;
on the fingers, see § 16.4;
with pen and paper, see 8.16.3 and 8.16.4;
binary, with Location Arithmatic as performed on a Chessboard, see § 11.12.2 and 11.12.3;
on a suan-pan or soroban, see § 17.6;
binary on a suan-pan, see § 17.16;
on a Pascaline, see § 14.1;
on a Comptometer, see § 14.2.

Multiplication and Division:

with Egyptian repeated doubling, see § 1.5 and 1.6;
with Mesopotamian tables and reciprocals, see § 2.8 and 2.9;
with Greek Ionic or Alexandrian numerals and tables, see § 3.6;
on or between the lines:
  on the Greek Salamis counting board, see § 3.5;
  on the Roman counting board, see § 4.4;
  the iron division, see § 5.7 and the golden division, see § 5.8 in Latin Europe to ca. 1000;
  Latin Europe from 1000 to 1600, see § 8.17.5 and 8.17.6, also doubling and halving § 8.17.4
with the erasing method:
  on a Hindu dust abacus, see § 6.8;
  on an Arabic dust abacus, see § 7.8, 7.9 and 7.10, also doubling and halving;
  Latin Europe from 1000 to 1600, see § 8.16.6 and 8.16.7;
With the crossing out or galley method:
  with Arabian pen and paper method, see § 7.10;
  Latin Europe from 1000 to 1600, see § 8.17.5;
With the gelosia method:
  India, see § 6.10;
  Latin Europe from 1000 to 1600, see § 8.17.5;
With pen and paper:
  Latin Europe from 1000 to 1600, see § 8.16.5 and 8.16.6, also doubling and halving;
with Napier's bones, see § 11.5 and 11.6;
with The High-speed Promptuary for Multiplication, see § 11.11;
with the Location Arithmatic as performed on a Chessboard, see § 11.12.4 and 5;
with Genaille-Lucas' rods, see § 13.1 and 13.2;
on the fingers, see § 16.7;
with logarithms, see § 10.1;
with a slide rule, see § 10.4 to 10.12;
on Schickard's calculator, see § 12.3 and § 12.6;
on Morland's calculator, see § 12.5;
on a Leibniz' calculator, see § 14.2.1.1;

on a CURTA, see § 14.2.1.5;
on a Odhner, Brunsviga and the like, see § 14.3.3;
on a millionaire, see § 14.4.4;
on a suan-pan, see § 17.7, 17.8 and 17.9;
on a soroban, see § 18.3 and 18.4.

Extracting Roots (now 8 different ways are known to me):

1) on a calculator; enter the correct data and read the result;

2) with a logarithm table (all powers), see § 9.7;
exactness depends on the number of decimals of the table, example:

$x = \sqrt[2]{22}$ or $x = 22^{½}$ $\log x = \log 22^{½} = ½\log 22$

log22 = 1,34242 looked up in a logarithm table (characteristic = 1, mantissa = 34242)

½ log22 = 0,67121 x = 4,6904 again looked up in the logarithm table.

On the same way also $\sqrt[¾]{11}$ etc.

3) on a slide rule provided with a L scale (all powers)

K	8	10.64	27	103.16	10	x^3
A	4	4.84	9	21.99	10	x^2
B	4	4.84	9	21.99	10	x^2
C	2	2.2	3	4.69	10	x
D	2	2.2	3	4.69	10	x
L	.301	.342	.477	.671	(mantissa) log$x$	

$x = \sqrt[2]{22}$ look for 22 on the D scale, read mantissa .342 on L and write the characteristic 1 in front of it = 1.342, than divide 1.342 by 2 = 0.671 (the result has 1 figure in front of the decimal point), read on D 4.69. It works very quick, but is less accutate.

on the LL scales of a Darmstadt slide rule, one can calculate powers and logarithms of anny base, see § 10.8;

4) with Ãryabhata's method on his sand-abacus for the square and cube root see § 6.9;
in Latin Europe to 1600 with pen and paper with the crossing out method, see § 8.17.7;

5) with Napier's rods, see § 11.7 for the square root
and § 11.8 for the cube root;

6) with Location Arithmatic as performed on a Chessboard, see § 11.12.5;

7) on a calculator as the CURTA, see § 14.2.1.5;
according the method of Töpler (based on stopping a arithmetic series on anny place:
e.g.: 1 + 3 + 5 + 7 = 16, the sum 16 is the square of the number of terms)

8) on a suan-pan, see § 17.10, 17.11, 17.3 and 17.14 for the second power
and § 17.12 for the third power.

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